(Due Tuesday 03/26/2019 right before the class)

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(Your homework shall be stapled if it contains multiple pages.)

SPRING/2019/MA526: HOMEWORK 6

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Total points: 20

Q1 ($2+2+2$ points) An exponential random variable X with parameter λ, for $\lambda \in (0, \infty)$ has the following cumulative distribution function

$$F(x) = \begin{cases} 0 & \text{if } x \le 0 \\ 1 - e^{-\lambda x} & \text{if } x > 0 \end{cases}$$

Find the PDF, Find the expectation and variance of X.

`Answer:` The PDF $f(x) = F'(x) = \lambda e^{-\lambda x}$ for $x > 0$ and $f(x) = 0$ for $x \le 0$.

Now we compute the expectation and variance.

$$\mathbb{E}[X] = \int_0^\infty x\lambda e^{-\lambda x} dx = -\int_0^\infty x d\big(e^{-\lambda x}\big)$$

then using integration by parts, we have

$$\mathbb{E}[X] = \left(-xe^{-\lambda x}\Big|_{x=0}^{x=+\infty}\right) + \int_0^\infty e^{-\lambda x} dx = \int_0^\infty e^{-\lambda x} dx = \frac{1}{\lambda}\int_0^\infty f(x)dx = \frac{1}{\lambda}.$$

Similarly,

$$\mathbb{E}[X^2] = \int_0^\infty x^2 \lambda e^{-\lambda x} dx = -\int_0^\infty x^2 d\big(e^{-\lambda x}\big) = \left(-x^2 e^{-\lambda x}\Big|_{x=0}^{x=+\infty}\right) + \int_0^\infty e^{-\lambda x} d(x^2) = 2\int_0^\infty xe^{-\lambda x} dx$$

Notice that

$$\int_0^\infty xe^{-\lambda x} dx = \frac{1}{\lambda}\int_0^\infty x\lambda e^{-\lambda x} dx = \frac{1}{\lambda}\mathbb{E}[X] = \frac{1}{\lambda^2} \Rightarrow \mathbb{E}[X^2] = \frac{2}{\lambda^2}$$

so

$$\text{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}.$$

Q2 (4pt) Suppose that airplane engines operate independently and fail with probability equal to 0.4. Assuming that a plane makes a safe flight if at least one-half of its engines run, determine whether a 4-engine plane or a 2- engine plane has the higher probability for a successful flight.

`Answer:` For a 4-engine plane, a successful flight means that during the flight, there are at least two engines run and its probability is equal to

$$\binom{4}{2} 0.6^2 0.4^2 + \binom{4}{3} 0.6^3 0.4^1 + \binom{4}{4} 0.6^4 = 82.08\%$$

For a 2-engine plane, a successful flight means that during the flight, there are at least 1 engine run and its probability is equal to

$$\binom{2}{1} 0.6 \times 0.4 + \binom{2}{2} 0.6^2 = 84\%$$

It follows that **a 2- engine plane has the higher probability for a successful flight.**

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Q3 (3pt) A safety engineer claims that only 40% of all workers wear safety helmets when they eat lunch at the workplace. Assuming that this claim is right, find the probability that 4 of 6 workers randomly chosen will be wearing their helmets while having lunch at the workplace.

`Answer:` The desired probability is

$$\binom{6}{4}0.4^4 \times 0.6^2 = 0.13824\,.$$

Q4 (3pt) Find the probability that a person flipping a coin gets the third head on the seventh flip.

`Answer:` To get the third head on the seventh flip means that the person get the head on the seventh flip and get the other two heads in the previous six flips. So the probability should be

$$\binom{6}{2}\left(\frac{1}{2}\right)^2\left(1-\frac{1}{2}\right)^4 \times \frac{1}{2} = 15 \times \left(\frac{1}{2}\right)^7 = \frac{15}{128} = 0.1171875.$$

Q5 (2+2 pt) Let X be a binomial random variable with parameters $(6, 0.4)$.

(1) Find the third moment $\mathbb{E}[X^3]$.

(2) Let Y be a Bernoulli(1/2) random variable that is independent of X. Find the probability

$$\mathbb{P}\big(X + Y = 5\big)\,.$$

`Answer:` (1)

$$\begin{aligned}\mathbb{E}[X^3] &= \sum_{k=0}^{6} k^3 \binom{6}{k} 0.4^k 0.6^{6-k} \\ &= 0 + 1 \times 6 \times 0.4 \times 0.6^5 + 2^3 \times 15 \times 0.4^2 \times 0.6^4 + 3^3 \times 20 \times 0.4^3 \times 0.6^3 + 4^3 \times 15 \times 0.4^4 \times 0.6^2 \\ &\qquad + 5^3 \times 6 \times 0.4^5 \times 0.6^1 + 6^3 \times 1 \times 0.4^6 \\ &= 24.48\end{aligned}$$

(2) As Y only takes two values 0, 1,

$$\mathbb{P}\big(X+Y=5\big) = \mathbb{P}\big(\{X+1=5, Y=1\} \cup \{X+0=5, Y=0\}\big) = \mathbb{P}\big(X=4, Y=1\big) + \mathbb{P}\big(X=5, Y=0\big)$$

then using independence, we write

$$\mathbb{P}\big(X=4, Y=1\big) = \mathbb{P}\big(X=4\big)\mathbb{P}\big(Y=1\big) = \frac{1}{2}\mathbb{P}\big(X=4\big) = \frac{1}{2}\binom{6}{4}0.4^4 0.6^2 = 0.06912$$

and

$$\mathbb{P}\big(X=5, Y=0\big) = \mathbb{P}\big(X=5\big)\mathbb{P}\big(Y=0\big) = \frac{1}{2}\mathbb{P}\big(X=5\big) = \frac{1}{2}\binom{6}{5}0.4^5 0.6^1 = 0.018432$$

we have then

$$\mathbb{P}\big(X+Y=5\big) = 0.087552.$$