HW 3 solution

Q1

(a) $\binom{4}{3}\left(\frac{1}{2}\right)^3 \frac{1}{2} = 4 \times \frac{1}{16} = \frac{1}{4} = 25\%$

(b) $\left(\frac{1}{2}\right)^4 = \frac{1}{16}$

Q2 we know $\mathbb{P}$roba(test says Yes | people do have allergy) = 75%

Proba("Yes" | no allergy) = 10%

& proba(a generic individual has allergy) = 1%

we want proba(Amy has allergy | "Yes") which is equal to

$$\frac{\mathbb{P}(\text{Amy has allergy \& "Yes"})}{\mathbb{P}(\text{"Yes"})} = \frac{\mathbb{P}(\text{Yes} \mid \text{Amy has allergy})\,\mathbb{P}(\text{Amy has allergy})}{\mathbb{P}(\text{Yes} \mid \text{allergy})\,\mathbb{P}(\text{allergy}) + \mathbb{P}(\text{Yes} \mid \text{no allergy})\,\mathbb{P}(\text{no allergy})}$$

$$= \frac{75\% \times 1\%}{75\% \times 1\% + 10\% \times 99\%} = \frac{75}{75 + 990} = \frac{5}{71} \approx \underline{0.0704} \text{ or } \underline{7.04\%}$$

Bayes' rule

Q3 (a) Not a pmf : because $f(0) + f(1) + f(3) = \frac{1}{4}(4 + 5 + 13) > 1$.

(b) C satisfies the equation $\int_0^2 C\sqrt{x}\,dx = 1$

$\Rightarrow C = \left(\int_0^2 \sqrt{x}\,dx\right)^{-1}$ we compute $\int_0^2 \sqrt{x}\,dx = \left(\frac{2}{3} x^{\frac{3}{2}}\right)\Big|_{x=0}^{x=2}$

$= \frac{2}{3} \times 2 \times \sqrt{2} = \frac{4\sqrt{2}}{3}$

So $C = \frac{3}{4\sqrt{2}} = \boxed{\frac{3\sqrt{2}}{8}}$

(c) From (b) $g(x) = \begin{cases} \frac{3}{8}\sqrt{2x} & 0 < x < 2 \\ 0 & \text{elsewhere} \end{cases}$ $\Rightarrow$ corresponding CDF $F(x) = \int_{-\infty}^x g(y)\,dy = \begin{cases} 0 & x \le 0 \\ \int_0^x \frac{3}{8}\sqrt{2y}\,dy & 0 < x < 2 \\ 1 & x \geq 2 \end{cases}$

or $F(x) = \begin{cases} 0 & x \le 0 \\ c\int_0^x \sqrt{y}\,dy & 0 < x < 2 \\ 0 & x \ge 2 \end{cases}$ with $c = \frac{3}{8}\sqrt{2}$

For $0 < x < 2$, $c\int_0^x \sqrt{y}\,dy = c \times \frac{2}{3} y^{\frac{3}{2}} \Big|_{y=0}^{y=x}$

$= \frac{2}{3} c\, x^{\frac{3}{2}} \underset{c=\frac{3}{8}\sqrt{2}}{=} \frac{2}{3} \times \frac{3}{8}\sqrt{2}\, x^{\frac{3}{2}} = \frac{\sqrt{2}}{4} x^{\frac{3}{2}}$

That's,

$$F(x) = \begin{cases} 0 & x \le 0 \\ \frac{\sqrt{2}}{4} x^{\frac{3}{2}} & 0 < x < 2 \\ 1 & x \ge 2 \end{cases}$$

Q4 (1) $\mathbb{P}(X^2 \le 1.21) = \mathbb{P}(-1.1 \le X \le 1.1) = \int_{-1.1}^{1.1} f(x)\,dx$

$= \int_0^1 x\,dx + \int_1^{1.1} (2-x)\,dx$

$= \frac{1}{2} + \left(2x - \frac{x^2}{2}\right)\Big|_{x=1}^{x=1.1}$

$= \frac{1}{2} + 0.095 = 0.595$

(2) we first find the pmf $f(x)$ of T:

$f(1) = \frac{1}{4}$ $\quad f(3) = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$ $\quad f(5) = \frac{3}{4} - \frac{1}{2} = \frac{1}{4}$ $\quad f(7) = 1 - \frac{3}{4} = \frac{1}{4}$

$f(x) = 0$ if x is not in $\{1, 3, 5, 7\}$.

So (i) $\mathbb{P}(T=5) = f(5) = \frac{1}{4}$

(ii) $\mathbb{P}(1.4 \le T < 6) = f(3) + f(5) = \frac{1}{2}$

End