solution to hw14 questions, MA526

`Exercises 10.2, 10.4, 10.8, 10.12 and 10.20 in our textbook`

Q1: (1) Type-I error occurs if one rejects H_0 when it is true. If SHE commits a type I error by erroneously concluding that the training course is ineffective. So

H_0 should be "**training course is effective** "

(2) Type-II error occurs if one accepts H_0 when it is false. If SHE commits a type II error by erroneously concluding that the training course is effective. So

H_0 should be "**training course is effective** "

Q2: A fabric manufacturer believes that the proportion of orders for raw material arriving late is $p = 0.6$. If a random sample of 10 orders shows that 3 or fewer arrived late, the hypothesis that $p = 0.6$ should be rejected in favor of the alternative $p < 0.6$. Use the binomial distribution.

(a) Find the probability of committing a type I error if the true proportion is $p = 0.6$.

(b) Find the probability of committing a type II error for the alternatives $p = 0.3, p = 0.4$, and $p = 0.5$.

Answer: (a) Assuming $H_0 : p = 0.6$,

$$\texttt{Prob}\big(X \leq 3\big)$$

with $X \sim \texttt{Binomial}(10, 0.6)$, so the above probability is equal to

$$\binom{10}{0}0.4^{10} + \binom{10}{1}0.4^9 0.6 + \binom{10}{2}0.4^8 0.6^2 + \binom{10}{3}0.4^7 0.6^3 = 0.0547618816.$$

(b) Let β denote the required probability:

(1) for $p = 0.3$,

$$\begin{aligned}\beta &= \texttt{Prob}\big(X > 3\big) \qquad \text{with } X \sim \texttt{Binomial}(10, 0.3)\\ &= 1 - \texttt{Prob}\big(X \leq 3\big) = 1 - \left\{\binom{10}{0}0.7^{10} + \binom{10}{1}0.7^9 0.3 + \binom{10}{2}0.7^8 0.3^2 + \binom{10}{3}0.7^7 0.3^3\right\}\\ &= 0.3503892816\end{aligned}$$

(2) for $p = 0.4$,

$$\begin{aligned}\beta &= \texttt{Prob}\big(X > 3\big) \qquad \text{with } X \sim \texttt{Binomial}(10, 0.4)\\ &= 1 - \texttt{Prob}\big(X \leq 3\big) = 1 - \left\{\binom{10}{0}0.6^{10} + \binom{10}{1}0.6^9 0.4 + \binom{10}{2}0.6^8 0.4^2 + \binom{10}{3}0.6^7 0.4^3\right\}\\ &= 0.6177193984\end{aligned}$$

(3) for $p = 0.5$,

$$\begin{aligned}\beta &= \texttt{Prob}\big(X > 3\big) \qquad \text{with } X \sim \texttt{Binomial}(10, 0.5)\\ &= 1 - \texttt{Prob}\big(X \leq 3\big) = 1 - \left\{\binom{10}{0}0.5^{10} + \binom{10}{1}0.5^{10} + \binom{10}{2}0.5^{10} + \binom{10}{3}0.5^{10}\right\}\\ &= 0.828125\end{aligned}$$

Q3: In Relief from Arthritis published by Thorsons Publishers, Ltd., John E. Croft claims that over 40% of those who suffer from osteoarthritis receive measurable relief from an ingredient produced by a particular species of mussel found off the coast of New Zealand. To test this claim, the mussel extract is to be given to a group of 7 osteoarthritic patients. If 3 or more of the patients receive relief, we shall not reject the null hypothesis that $p = 0.4$; otherwise, we conclude that $p < 0.4$.

(a) Evaluate α, assuming that $p = 0.4$.

(b) Evaluate β for the alternative $p = 0.3$.

Answer: We use Binomial distribution!

$$\begin{aligned}\alpha &= \text{Probability of type I error} = \mathbb{P}\big(X < 3 \text{ under } p = 0.4\ \big)\\ &= \binom{7}{0}0.6^7 + \binom{7}{1}0.6^6 0.4 + \binom{7}{2}0.6^5 0.4^2 = 0.4199\\ &\text{and}\\ \beta &= \text{Probability of type II error} = \mathbb{P}\big(Y \geq 3 \text{ under } p = 0.3\ \big)\\ &= 1 - \left\{\binom{7}{0}0.7^7 + \binom{7}{1}0.7^6 0.3 + \binom{7}{2}0.7^5 0.3^2\right\} = 0.3529\end{aligned}$$

where $X \sim \texttt{Binomial}(7, 0.4)$ and $Y \sim \texttt{Binomial}(7, 0.3)$.

Q4: A random sample of 400 voters in a certain city are asked if they favor an additional 4% gasoline sales tax to provide badly needed revenues for street repairs. If more than 220 but fewer than 260 favor the sales tax, we shall conclude that 60% of the voters are for it.

(a) Find the probability of committing a type I error if 60% of the voters favor the increased tax.

(b) What is the probability of committing a type II error using this test procedure if actually only 48% of the voters are in favor of the additional gasoline tax?

Answer: We use normal approximation of Binomial distribution!

(a) Under $p = 0.6$, the binomial distribution $\texttt{Binomial}(400, 0.6)$ is approximately normal with mean 240 and standard deviation $\sqrt{400 \times 0.6 \times 0.4} \approx 9.8$, so

$$\begin{aligned}\alpha &= \mathbb{P}\big(X < 220 \textit{ or } X > 260\big) \qquad \text{with } X \sim \texttt{Binomial}(400, 0.6)\\ &= \mathbb{P}\big(\frac{X - 240}{9.8} < \frac{-20}{9.8}\big) + \mathbb{P}\big(\frac{X - 240}{9.8} > \frac{20}{9.8}\big) \approx 2\Phi\big(-20/9.8\big)\end{aligned}$$

where Φ is the standard normal CDF. Checking the normal table, we get $\alpha \approx 0.04136$.

(b) Assuming $p = 0.48$ now, $\texttt{Binomial}(400, 0.48)$ is approximately normal with mean 192 and standard deviation 9.99

$$\begin{aligned}\beta &= \mathbb{P}\big(220 < X < 260\big) \qquad \text{with } X \sim \texttt{Binomial}(400, 0.48)\\ &= \mathbb{P}\big(\frac{28}{9.99} < \frac{X - 192}{9.99} < \frac{68}{9.99}\big) \approx \Phi(6.81) - \Phi(2.80) \approx 1 - 0.99744 = 0.00256\,.\end{aligned}$$

`One can also do computation taking the continuity correction into her/his consideration.....`

Q5: A random sample of 64 bags of white cheddar popcorn weighed, on average, 5.23 ounces with a standard deviation of 0.24 ounce. Test the hypothesis that $\mu = 5.5$ ounces against the alternative hypothesis, $\mu < 5.5$ ounces, at the 0.05 level of significance.

Answer: $H_0 : \mu = 5.5$ and $H_1 : \mu < 5.5$. (This is a one-tailed test)

We know from the sample that $\overline{x} = 5.23$, $s = 0.24$ and we use sample average $\overline{X}$ as our test statistic. Here

$$\overline{X} = \frac{1}{64}(X_1 + X_2 + \ldots + X_{64})$$

with $X_1, X_2, \ldots, X_{64}$ i.i.d. sampled from our population distribution. In this problem, we do not know the exact value of the population standard deviation and we are not informed of whether the population distribution is approximately normal or not. So we claim roughly that $\frac{\overline{X}-\mu}{s/\sqrt{n}}$ is approximately standard normal.

As we are in the one-tailed test, the critical region should be $(-\infty, -1.65)$, as $\mathbb{P}\big(Z < -1.65\big) = 0.05$ with Z standard normal. While our sample evidence shows that under H_0:

$$\frac{\overline{x} - \mu}{s/\sqrt{n}} = \frac{5.23 - 5.5}{0.24/\sqrt{64}} = -9 < -1.65$$

so we reject H_0 in favor of H_1.

One can also compute the P-value, then argue....