solution to hw13 questions, MA526

`Exercises 9.36, 9.40, 9.48, 9.54, 9.58 and 9.72 in our textbook`

Q1: We need to estimate $\mu_A - \mu_B$, where μ_A is the population mean tensile strength of *brand A* and μ_B is the population mean tensile strength of *brand B*. For both brands, we sampled $n = 50$ pieces with sample average, sample standard deviation given by

$$\overline{x}_A = 78.3\,, \quad s_A = 5.6 \qquad \text{and} \qquad \overline{x}_B = 87.2\,, \quad s_B = 6.3\,.$$

We do not know the values of population variance, which are likely unequal in view of the sample evidence. We shall use the so-called *Satterthwaite approximation*: the point estimate is $\overline{x}_A - \overline{x}_B = -8.9$. The $100(1-\alpha)\%$ confidence interval, with $\alpha = 0.05$, is given by

$$\overline{x}_A - \overline{x}_B \pm t_{0.025}\sqrt{\frac{s_A^2 + s_B^2}{n}}$$

$t_{0.025}$ is the t-value associated to the t-distribution with degree of freedom v, v being the nearest integer to the following quantity

$$\frac{(s_A^2/n + s_B^2/n)^2}{\left[(s_A^2/n)^2/(n-1)\right] + \left[(s_B^2/n)^2/(n-1)\right]} \approx 96.6712020268$$

that is, we take $v = 97$, then we take $t_{0.025} = 1.99$. In the t-table, we do not see the row corresponding to 97, we only have row 60 and row 120, their intersection with the column 0.025 are 2.000 and 1.980, so we take the value in between as our value for $t_{0.025}$.

We compute $\dfrac{s_A^2 + s_B^2}{n} = \dfrac{5.6^2 + 6.3^2}{50} \approx 1.19$, then the desired confidence interval is given by

$$-8.9 \pm (1.99 \times 1.19) = (-11.27, -6.53)\,.$$

Q2: Let $\overline{X}_1$ and $\overline{X}_2$ denote the statistics (sample average) for two kinds of seedlings: no Nitrogen and Nitrogen. As the populations are assumed to be normal with equal variances, we assume the variance is equal to σ^2, then

$$\frac{\overline{X}_1 - \overline{X}_2 - (\mu_1 - \mu_2)}{\sigma/\sqrt{n/2}} \sim \mathcal{N}(0,1)$$

where $\mu_1 - \mu_2$ is what we want to estimate. It is also clear that $\dfrac{(n-1)S_1^2 + (n-1)S_2^2}{\sigma^2}$ is a chi-squared random variable with degree of freedom $2n-2$, where S_1^2, S_2^2 stand for the sample variance statistics. Then

$$T := \left(\frac{\overline{X}_1 - \overline{X}_2 - (\mu_1 - \mu_2)}{\sigma/\sqrt{n/2}}\right) \Big/ \sqrt{\frac{(n-1)S_1^2 + (n-1)S_2^2}{\sigma^2(2n-2)}} = \frac{\overline{X}_1 - \overline{X}_2 - (\mu_1 - \mu_2)}{\sqrt{\dfrac{S_1^2 + S_2^2}{n}}}$$

is a t-random variable with degree of freedom $2n - 2 = 18$. With $\overline{x}_1 - \overline{x}_2$ as the point estimate for $\mu_1 - \mu_2$, we have the 95% confidence interval

$$\left(\overline{x}_1 - \overline{x}_2\right) \pm t_{0.025}\sqrt{\frac{s_1^2 + s_2^2}{n}}$$

and we can find the values for $\overline{x}_1 - \overline{x}_2$ and $s_1^2 + s_2^2$ from the given data:

$$\overline{x}_1 = 0.399, \quad s_1^2 = 0.00529888888889 \quad \textit{and} \qquad \overline{x}_2 = 0.565 \quad s_2^2 = 0.0348722222222$$

and $t_{0.025}$, (for the degree of freedom =18), is equal to 2.101, so the required confidence interval is given by

$$(0.399 - 0.565) \pm \sqrt{(0.00529888888889 + 0.0348722222222)/10} = (-0.299, -0.0328)$$

Q3: In the exactly same argument, we get the 95% confidence interval for $\mu_A - \mu_B$ to be

$$(\overline{x}_A - \overline{x}_B) \pm t_{0.025}\sqrt{\frac{s_A^2 + s_B^2}{20}}$$

and $t_{0.025} \approx 2.02$ as degree of freedom is 38 now. So the required 95% confidence interval for $\mu_A - \mu_B$ is

$$(1.3814, 3.4986)$$

by plugging the values for $\overline{x}_A, \overline{x}_B, s_A^2, s_B^2$. So the automotive company should adopt type A battery.

Q4: We know from the problem that $n = 500$, $\widehat{p} = 485/500 = 0.97$. As we are looking for the 90% confidence interval, we find in the normal table $z_{0.05} \approx 1.65$, then the required interval is

$$0.97 \pm 1.65\sqrt{\frac{0.97 \times 0.03}{500}} = 0.97 \pm 0.01258767254 = (0, 957, 0.983)\,.$$

Q5: By Theorem 9.5 in the textbook or some result from our lecture, we need $n \geq \frac{z_{\alpha/2}^2}{4\varepsilon^2}$, if we use $\widehat{p}$ as an estimate of p and be at least $100(1-\alpha)\%$ confident to say the error $|\widehat{p} - p|$ will not exceed ε.

So in our problem, we need

$$n \geq \frac{z_{0.025}^2}{4(0.02)^2} = \frac{1.96^2}{4(0.02)^2} = 49^2 = 2401\,.$$

Q6: 98% confidence interval for σ^2 is given by

$$\frac{(n-1)s^2}{\chi_{\alpha/2}^2} < \sigma^2 < \frac{(n-1)s^2}{\chi_{1-\alpha/2}^2}$$

with $\alpha = 0.02$. $\chi_{\alpha/2}^2 = \chi_{0.01}^2 = 36.191$ and $\chi_{1-\alpha/2}^2 = \chi_{0.99}^2 = 7.633$ [here the degree of freedom =19] so the interval is

$$8.39987842 < \sigma^2 < 39.8270666841$$