Solution to Mid-term 2

Q1. the random variable Z in our question only takes two values 0 and 1, so it is a Bernoulli random variable with success probability p to be determined. And we know that

$$1-p = \mathbf{P}(Z=0) = \mathbf{P}(\max\{X,Y\}=0) = \mathbf{P}(X=0 \quad \textit{and} \quad Y=0) = \mathbf{P}(X=0)\times\mathbf{P}(Y=0) = 1/4$$

so $Z \sim \texttt{Bernoulli}(3/4)$. Its probability mass function is given by

$f(x) = 3/4$, if $x = 1$;

$f(x) = 1/4$, if $x = 0$;

$f(x) = 0$ else.

It is clear that $\mathbf{E}[Z] = 3/4$ and $\mathbf{Var}[Z] = 3/16$

Q2. (1) $c > 0$ should satisfy

$$c\int_0^1\int_0^1 x^2 y\,dx\,dy = 1$$

which implies

$$1 = c\left(\int_0^1 x^2\,dx\right)\left(\int_0^1 y\,dy\right) = c\times\frac{1}{3}\times\frac{1}{2}$$

so $c = 6$.

(2) We first find the marginals:

$$f_X(x) = \int_0^1 f(x,y)\,dy = 3x^2$$

for $0 < x < 1$ and

$$f_Y(y) = \int_0^1 f(x,y)\,dx = 2y$$

for $0 < y < 1$.

That is, we have $f(x,y) = f_X(x)f_Y(y)$. So X, Y are **independent.**

(3) Due to independence from (2), we have $\mathbf{Var}(X^2+Y^2) = \mathbf{Var}(X^2) + \mathbf{Var}(Y^2)$.

We know from the definition that $\mathbf{Var}(X^2) = \mathbf{E}(X^4) - \Big[\mathbf{E}(X^2)\Big]^2$

Using the marginals, we get

$$\mathbf{E}(X^4) = \int_0^1 3x^2x^4\,dx = 3/7 \quad \textit{and} \quad \mathbf{E}(X^2) = \int_0^1 3x^2x^2\,dx = 3/5$$

so $\mathbf{Var}(X^2) = \mathbf{E}(X^4) - \Big[\mathbf{E}(X^2)\Big]^2 = \frac{3}{7} - (3/5)^2 = 12/175$.

Similarly, $\mathbf{Var}(Y^2) = \mathbf{E}(Y^4) - \Big[\mathbf{E}(Y^2)\Big]^2$ is equal to

$$\int_0^1 2yy^4\,dy - \left(\int_0^1 2yy^2\,dy\right)^2 = (2/6) - (2/4)^2 = 1/12.$$

so $\mathbf{Var}(X^2+Y^2) = \mathbf{Var}(X^2) + \mathbf{Var}(Y^2) = 12/175 + 1/12 = 319/2100 \approx 0.1519047619$.

Q3. (1) we have $0.47831 = \Phi(2.02) - \Phi(k)$, where Φ stands for the standard normal CDF. We find from the normal table that $\Phi(2.02) = 0.97831$, so $\Phi(k) = 0.5$, so $k = 0$.

(2) $X \sim \texttt{Binomial}(100, 1/2)$. Its mean is 50 and its variance is 25, standard deviation is 5. We know from normal approximation that X is close to a normal distribution $\mathcal{N}(50, 25)$.
Then

$$\mathbf{P}(50 < X < 75) = \mathbf{P}(50-50 < X-50 < 75-50) = \mathbf{P}(0 < X-50 < 25) = \mathbf{P}(0 < (X-50)/5 < 5)$$

and $(X-50)/5$ is close to the standard normal, so that $\mathbf{P}(50 < X < 75) \approx \Phi(5) - \Phi(0) \approx 1 - 0.5 = 0.5$. You could also do normal approximation with the continuity correction.

Q4. $\mu = 4 \times 0.2 + 5 \times 0.4 + 6 \times 0.3 + 7 \times 0.1 = 5.3$. So $\mathbf{E}(\bar{X}) = \mu = 5.3$.

And $\sigma^2 = \mathbf{E}(X^2) - 5.3^2 = (4^2 \times 0.2 + 5^2 \times 0.4 + 6^2 \times 0.3 + 7^2 \times 0.1) - 5.3^2 = 0.81$, and $\sigma = 0.9$
So with $n = 36$, $\mathbf{Var}(\bar{X}) = \frac{\sigma^2}{n} = 0.81/36 = 9/400 = 0.0225$.

We know that the sample average $\bar{X}$ is close to the normal distribution with mean 5.3 and variance $9/400 = (3/20)^2$. Then

$$\mathbf{P}(\bar{X} \le 5.5) = \mathbf{P}((\bar{X} - 5.3)/(3/20) \le (5.5 - 5.3)/(3/20)) \approx \Phi(4/3) \approx \Phi(1.33333) \approx 0.9082.$$

here Φ stands for the standard normal CDF.

Bonus 1. $Z = \min\{X, Y\}$ is a nonnegative random variable for sure. Fix $t > 0$, we have

$$\mathbf{P}(Z > t) = \mathbf{P}(\min\{X, Y\} > t) = \mathbf{P}(X > t \quad \textit{and} \quad Y > t) = \mathbf{P}(X > t)\mathbf{P}(Y > t)$$

since X, Y are independent. We know X is an exponential random variable with parameter 1 and Y is an exponential random variable with parameter $1/2$, thus

$$\mathbf{P}(Z > t) = \mathbf{P}(X > t)\mathbf{P}(Y > t) = e^{-t}e^{-t/2} = e^{-3t/2}$$

this tells us that Z is an exponential random variable with parameter $3/2$.

Bonus 2. If $X \sim \texttt{Poisson}(\lambda)$ and $Y \sim \texttt{Poisson}(\mu)$ are independent, then their sum is $\texttt{Poisson}(\mu + \lambda)$:

Indeed, it is clear that $X + Y$ is a random variable with values in $\{0, 1, 2, \ldots\}$. And for $n \in \{0, 1, 2, \ldots\}$, we have

$$\mathbf{P}(X+Y = n) = \sum_{k=0}^{n} \mathbf{P}(X = k, Y = n-k) = \sum_{k=0}^{n} \mathbf{P}(X = k)\mathbf{P}(Y = n-k) \quad (\textit{independence}!)$$

and $\mathbf{P}(X = k) = e^{-\lambda}\lambda^k/k!$ and $\mathbf{P}(Y = n - k) = e^{-\mu}\mu^{n-k}/(n-k)!$, then

$$\mathbf{P}(X + Y = n) = e^{-\lambda-\mu} \sum_{k=0}^{n} (\lambda^k/k!)(\mu^{n-k}/(n-k)!) = \frac{e^{-\lambda-\mu}}{n!} \sum_{k=0}^{n} n!(\lambda^k/k!)(\mu^{n-k}/(n-k)!)$$

Note that

$$\sum_{k=0}^{n} n!(\lambda^k/k!)(\mu^{n-k}/(n-k)!) = \sum_{k=0}^{n} \binom{n}{k} \lambda^k \mu^{n-k} = (\lambda + \mu)^n$$

implying

$$\mathbf{P}(X + Y = n) = e^{-\lambda-\mu}(\lambda + \mu)^n/n!$$

this is the PMF of $\texttt{Poisson}(\lambda + \mu)$. In this problem $\lambda = 1, \mu = 2$.