(Due Wednesday 10/24/2018 right before class)
&
(Your homework shall be stapled if it contains multiple pages.)

FALL/2018/MA526: HOMEWORK 9 SOLUTION

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Total points: 20.

Q1 (3pt) Find the PDF of X^3, where X is the standard normal random variable.

Answer: Let us denote by ϕ and Φ the standard normal PDF and CDF respectively: for every x

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

and

$$\Phi(x) = \int_{-\infty}^{x} \phi(y)\, dy\,.$$

Now we look for the PDF of X^3 with $X \sim \mathcal{N}(0,1)$: We start with its CDF, that is,

$$\mathbb{P}(X^3 \le t) = \mathbb{P}(X \le t^{1/3}) = \Phi(t^{1/3})$$

so that the PDF of X^3, which we denote by f, is given by

$$f(t) = \frac{d}{dt}\mathbb{P}(X^3 \le t) = \frac{d}{dt}\Phi(t^{1/3}) = \phi(t^{1/3})\frac{d}{dt}(t^{1/3}) = \phi(t^{1/3})\frac{1}{3}t^{-2/3}$$

then using the expression for ϕ, we get

$$f(t) = \frac{1}{3\sqrt{2\pi}} e^{-\frac{t^{2/3}}{2}} t^{-2/3}$$

or written as $\frac{1}{3\sqrt{2\pi}} \exp\Big(-\frac{t^{2/3}}{2}\Big) t^{-2/3}$.

Note that $f(0) = +\infty$, so when we write down the expression of the PDF of X^3, we can do as follows:

$$f(t) = \frac{1}{3\sqrt{2\pi}} \exp\Big(-\frac{t^{2/3}}{2}\Big) t^{-2/3} \quad t \neq 0\,.$$

and you can put any real value for $f(0)$.

Student shall get the full points even if they have not specified the issue with $f(0)$.

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Q2 (6pt) In a human factor experimental project, it has been determined that the reaction time of a pilot to a visual stimulus is normally distributed with a mean of 1/2 second and standard deviation of 2/5 second.

(a) What is the probability that a reaction from the pilot takes more than 0.3 second?

(b) What reaction time is that which is exceeded 95% of the time?

Answer: Denote by X the reaction time of a pilot to a visual stimulus, then $X \sim \mathcal{N}(0.5, 0.16)$. Note that the variance is $(2/5)^2 = 0.16$. Then we know that

$$N = \frac{X - 0.5}{0.4} \quad \text{is a standard normal random variable}$$

(a)

$$\mathbb{P}(X > 0.3) = \mathbb{P}\left(\frac{X - 0.5}{0.4} > \frac{0.3 - 0.5}{0.4}\right) = \mathbb{P}(N > -0.5) = 1 - \Phi(-0.5)$$

here we use Φ to mean the standard normal CDF. Now we use the normal table and find $\Phi(-0.5) \approx 0.30854$ so $\mathbb{P}(X > 0.3) \approx 1 - 0.3085 = 0.6915$.

(b) We look for some approximate value x such that

$$\mathbb{P}(X \le x) \approx 0.95$$

That is,

$$\mathbb{P}(X \le x) = \mathbb{P}\left(\frac{X - 0.5}{0.4} \le \frac{x - 0.5}{0.4}\right) \approx 0.95$$

put $y = \frac{x-0.5}{0.4}$, we need $\Phi(y) \approx 0.95$. We find in the normal table that y should be between 1.64 and 1.65, so

$$1.64 < \frac{x - 0.5}{0.4} < 1.65$$

giving us $1.156 < x < 1.16$. `Any number in the range shall be regarded as a good answer.`

Q3 (6pt) The density function of the time T in minutes between calls to an electrical supply store is given by

$$f(x) = \begin{cases} \frac{1}{10} e^{-x/10}, & x \ge 0 \\ 0 & elsewhere \end{cases}$$

(a) What is the mean time between calls?

(b) What is the variance in the time between calls?

(c) What is the probability that the time between calls exceeds the mean?

Answer: (a) The expected value of an exponential random variable with parameter λ is $1/\lambda$, so the mean time between calls in this question is 10 minutes. Or you can compute the following integral

$$\int_0^\infty x \frac{1}{10} e^{-x/10}\, dx = 10\,.$$

(b) Let us compute the second moment of an exponential random variable with parameter λ:

$$\int_0^\infty x^2 \lambda e^{-\lambda x}\, dx = -\int_0^\infty x^2 d\left(e^{-\lambda x}\right) = -x^2 e^{-\lambda x}\Big|_0^{+\infty} + \int_0^\infty e^{-\lambda x} d(x^2) = \int_0^\infty e^{-\lambda x} d(x^2)$$

and

$$\int_0^\infty e^{-\lambda x} d(x^2) = \int_0^\infty 2x e^{-\lambda x} dx = \frac{2}{\lambda} \underbrace{\int_0^\infty x \lambda e^{-\lambda x} dx}_{mean!} = \frac{2}{\lambda} \times \frac{1}{\lambda} = \frac{2}{\lambda^2}$$

so the variance is equal to $\frac{2}{\lambda^2} - (\frac{1}{\lambda})^2 = \frac{1}{\lambda^2}$. In this question, the variance in the time between calls is 100 $(minutes)^2$. `[ FYI: The standard deviation is 10 minutes]`

(c) we look for

$$\mathbb{P}\big(\text{waiting time between calls} > 10 \text{ minutes}\big) = 1 - F(10)$$

where F is the CDF of an exponential random variable with parameter 1/10, so we know from previous lecture that for $t > 0$

$$1 - F(t) = e^{-t/10}$$

hence

$$\mathbb{P}\big(\text{waiting time between calls} > 10 \text{ minutes}\big) = 1 - F(10) = e^{-1} \approx 0.36787944117$$

Q4 (5pt) The lengths of time, in minutes, that 10 patients waited in a doctor's office before receiving treatment were recorded as follows: 5, 11, 9, 5, 10, 15, 6, 10, 5, and 10. Treating the data as a random sample, find

(a) the mean;

(b) the median;

(c) the mode;

(d) the standard deviation;

(e) the range.

Answer: (a) mean = 8.6

(b) re-order the sample as follows: 5,5,5,6,9,10,10,10,11,15. so the median is (9+10)/2 = 9.5.

(c) there are two sample modes: 5 and 10.

(d) sample variance is

$$\frac{1}{10-1}\Big((5-8.6)^2 \times 3 + (6-8.6)^2 + (9-8.6)^2 + (10-8.6)^2 \times 3 + (11-8.6)^2 + (15-8.6)^2\Big) \approx 10.9333333333$$

so the standard deviation is approximately $\sqrt{10.9333333333} \approx 3.30655913753$

(e) the range is the difference between the maximal value of the data and the minimal value within the data: so in this question, we have 15-5=10 as the range.