(Due Wednesday 10/17/2018 right before class)
&
(Your homework shall be stapled if it contains multiple pages.)

FALL/2018/MA526: HOMEWORK 8 SOLUTION

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Total points: 20.

Note to students: You may want to use the normal table that was distributed in the class. This table is also available on the instructor's teaching webpage.

Q1 (2pt) Look up the normal table and find the value k such that

$$\mathbb{P}\big(-1.44 < Z < k\big) = 0.24466\,,$$

where $Z \sim \mathcal{N}(0,1)$ is a standard normal random variable.

Ans: We have $0.24466 = \mathbb{P}\big(-1.44 < Z < k\big) = \mathbb{P}\big(Z < k\big) - \mathbb{P}(Z \le -1.44) = \Phi(k) - \Phi(-1.44)$, where we write Φ for the CDF of a standard normal.

From the normal table, we get $\Phi(-1.44) = 0.07493$, so we get $\Phi(k) = 0.07493 + 0.24466 = 0.31959$. We shall try to find values that are close to this number in our normal table: We find

$$\Phi(-0.46) = 0.32276 \quad \text{and} \quad \Phi(-0.47) = 0.31918\,.$$

So it is ok to pick k from the interval $(-0.47, -0.46)$. Any number in this interval shall be regarded as a good answer.

Q2 (6pt) Given a normal distribution with mean 30 and variance 36, find

(a) the normal curve area to the right of $x = 17$;

(b) the normal curve area between $x = 32$ and $x = 41$;

(c) the value of x that has 80% of the normal curve area to the left.

Ans: (a) Let us write $X \sim \mathcal{N}(30, 36)$. The probability in (a) can be expressed as $\mathbb{P}\big(X \ge 17\big)$, which is equal to

$$\mathbb{P}\big(X \ge 17\big) = \mathbb{P}\Big(\frac{X-30}{6} \ge \frac{17-30}{6}\Big)\,.$$

we know that $\frac{X-30}{6}$ is a standard normal, so the above probability is equal to $1 - \Phi(-13/6)$.

Note that $-13/6 = -2.166666666$, so you can choose the value $\Phi(-2.16)$ or $\Phi(-2.17)$ from the normal table, which are 0.01539 and 0.01500 respectively.

Accordingly,

$$\mathbb{P}\big(X \ge 17\big) = 1 - \Phi(-13/6) = 0.98461 \quad \text{or} \quad 0.985$$

(b) We want $\mathbb{P}(32 < X < 41)$:

$$\mathbb{P}(32 < X < 41) = \mathbb{P}\Big(\frac{32-30}{6} \le \frac{X-30}{6} \le \frac{41-30}{6}\Big) \approx \mathbb{P}\Big(0.3333333 \le \frac{X-30}{6} \le 1.83333333333\Big) = \Phi(1.83) - \Phi(0.33)$$

we read from the normal table that $\Phi(1.83) - \Phi(0.33) = 0.96638 - 0.6293 = 33.708\%$

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(c) We want to find x such that $\mathbb{P}(X \le x) = 0.8$. Note that

$$0.8 = \mathbb{P}(X \le x) = \mathbb{P}\big(\frac{X-30}{6} \le \frac{x-30}{6}\big) = \Phi\big(\frac{x-30}{6}\big)\,.$$

from the normal table, we find

$$\Phi(0.84) = 79.955\% \quad and \quad \Phi(0.85) = 80.234\%$$

So we can set

$$0.84 \le \frac{x-30}{6} \le 0.85$$

or equivalently, $35.04 \le x \le 35.1$. Any number in this range shall be regarded as a good answer.

Q3 (4pt) A drug manufacturer claims that a certain drug cures a blood disease, on the average, 80% of the time. To check the claim, government testers use the drug on a sample of 100 individuals and decide to accept the claim if 75 or more are cured.

(a) What is the probability that the claim will be rejected when the cure probability is, in fact, 0.8?

(b) What is the probability that the claim will be accepted by the government when the cure probability is as low as 0.7

`Keyword: normal approximation, binomial distribution`

Answer: (a) Assuming the cure probability is 0.8, the number X of cured individuals (among the 100) is a random variable following the `binomial`(100, 0.8) distribution. Then,

$$\text{prob(the claim will be rejected)} = \text{prob}\big(X \le 74\big) = \sum_{k=0}^{74} \binom{100}{k} 0.8^k 0.2^{100-k}$$

one can find the exact value for the above sum. Here instead, we do normal approximation: When $X \sim$ `binomial`(n, p)

$$\text{prob}\big(X \le s\big) \approx \Phi\big(\frac{s - np}{\sqrt{np(1-p)}}\big)$$

and this approximation will be better when n is larger. In this question,

$$\text{prob}\big(X \le 74\big) \approx \Phi\big(-1.5\big) = 6.68\% \qquad (\#)$$

(When the steps are right, the final answer that is close to 6.68% shall be treated as a good answer. For example, it is ok to use 75 instead of 74 in (#).)

(b) Assuming the cure probability is 0.7, the number X of cured individuals (among the 100) is a random variable following the `binomial`(100, 0.7) distribution. Then,

$$\text{prob(the claim will be accepted)} = \text{prob}\big(X \ge 75\big) = 1 - \sum_{k=0}^{74} \binom{100}{k} 0.7^k 0.3^{100-k}$$

one can find the exact value for the above sum. Here instead, we do normal approximation: In this part,

$$\text{prob}\big(X \le 74\big) \approx \Phi\big(\frac{74-70}{\sqrt{21}}\big) \approx \Phi(0.87) \approx 19.215\%$$

So in this case, the probability that the claim will be accepted is close to 80.785%.

Q4 (4pt) The life, in years, of a certain type of electrical switch has an exponential distribution with an average life 2 years. If 100 of these switches are installed in different systems, what is the probability that at most 2 fail during the first year?

Ans: Recall that an exponential random variable with parameter $\lambda > 0$ has density function

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

Its expectation is equal to $1/\lambda$, as we computed in the class. So in this question, the life of this switch is an exponential random variable with parameter $1/2$ and its CDF is given by

$$F(x) = \begin{cases} 0 & x \leq 0 \\ 1 - e^{-x/2} & x > 0 \end{cases}$$

So Prob(any such switch fail during the first year)= $F(1) = 1 - e^{-1/2} \approx 0.39346934028$.

Now we want the probability that at most 2 fail during the first year: Note that the number of switches that fail during the first year is a binomial random variable with parameters $(100, p)$ with $p = F(1) = 1 - e^{-1/2} \approx 0.39346934028$.

The desired probability is equal to

$$\binom{100}{0} p^0 (1-p)^{100} + \binom{100}{1} p^1 (1-p)^{100-1} + \binom{100}{2} p^2 (1-p)^{98} = e^{-50} + 100 \times (1-e^{-1/2}) e^{-49.5} + 4950 \times (1-e^{-1/2})^2 e^{-49}$$

if you plug the above number in your calculator, you will probably get something displayed like

$$4.1449347e - 19,$$

which is close to 0.00000000000000000041449347. So the probability that at most 2 fail during the first year is close to zero. If a student writes down zero or something like the above small number, he/she is right if the steps towards this answer are correct.

Q5 (4pt) Let X be a lognormal$(0, 100)$ random variable, that is, X is a positive random variable and $\log X$ is a normal random variable with mean 0 and standard deviation 10. Denote by F the cumulative distributional function of X, then find the following values:

$$F(\log 10) \qquad \textit{and} \qquad F(0)\,.$$

It is enough for you to express the above values in term of the standard normal CDF. If you find their approximate value using the normal table, it will be nice.

Ans: $F(\log 10) = \mathbb{P}(X \leq \log 10) = \mathbb{P}(\frac{\log X}{10} \leq \frac{\log\log 10}{10}) = \Phi\Big(\frac{\log\log 10}{10}\Big)$, where Φ denotes the standard normal CDF. If students stop with this answer, that is good enough.

$F(0) = 0$, as X is a positive random variable.

A bit further: If you see log as the logarithm function with base 10, then $\log\log 10 = \log 1 = 0$, so $\Phi\Big(\frac{\log\log 10}{10}\Big) = \Phi(0) = 1/2$.

If you see log as the logarithm function with base e, then $\log\log 10 = \log 1 = 0.83403244524$, so $\Phi\Big(\frac{\log\log 10}{10}\Big) \approx \Phi(0.83) = 0.79673$.