(Due Wednesday 10/10/2018 right before class)
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(Your homework shall be stapled if it contains multiple pages.)

FALL/2018/MA526: HOMEWORK 7

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Total points: 20.

Q1 (4pt) Suppose X is a random variable with $\mathbb{E}[X] = 50$ and $\mathrm{Var}(X) = 25$. Calculate the following quantities.

(a) $\mathbb{E}[X^2]$ (b) $\mathbb{E}[2X + 10]$ (c) $\mathbb{E}[(X + 1)^2]$ (d) $\mathrm{Var}[-2X]$

`Ans:` (a) We know from the lecture that $\mathrm{Var}(X) = \mathbb{E}[X^2] - \big(\mathbb{E}[X]\big)^2$, so

$$\mathbb{E}[X^2] = \mathrm{Var}(X) + \big(\mathbb{E}[X]\big)^2 = 25 + 50^2 = 2525\,.$$

(b) $\mathbb{E}[2X + 10] = 2\mathbb{E}[X] + 10 = 110$.

(c)$\mathbb{E}[(X + 1)^2] = \mathbb{E}\big[X^2 + 2X + 1\big] = \mathbb{E}\big[X^2\big] + 2\mathbb{E}[X] + 1 = 2525 + 2 \times 50 + 1 = 2626$.

(d) $\mathrm{Var}[-2X] = 4\mathrm{Var}(X) = 100$.

Q2 (4pt) Compute the probability $\mathbb{P}\big(\mu - 2\sigma < X < \mu + 2\sigma\big)$, where X has the density function

$$f(x) = \begin{cases} 6x(1-x)\,, & 0 < x < 1\,, \\ 0 & elsewhere\,. \end{cases}$$

(The constant $\mu = \mathbb{E}[X]$ and $\sigma = \sqrt{\mathrm{Var}(X)}$.) Compare with the result given in the Chebyshev's theorem.

`Ans:` Let us first find out the values of μ, σ:

$$\mu = \mathbb{E}[X] = \int_0^1 x \times 6x(1-x)\,dx = \int_0^1 (6x^2 - 6x^3)\,dx = (2x^3 - \frac{3}{2}x^4)\big|_0^1 = 1/2$$

and

$$\mathbb{E}\big[X^2\big] = \int_0^1 x^2 \times 6x(1-x)\,dx = \int_0^1 (6x^3 - 6x^4)\,dx = (\frac{3}{2}x^4 - \frac{6}{5}x^5)\big|_0^1 = 3/10$$

so $\sigma^2 = \mathbb{E}\big[X^2\big] - \mu^2 = 1/20$, thus, $\sigma = \sqrt{1/20} = \sqrt{5}/10$. Then (**Note that** $0 < \frac{1}{2} - \frac{\sqrt{5}}{5} < \frac{1}{2} + \frac{\sqrt{5}}{5} < 1$)

$$\begin{aligned}\mathbb{P}\big(\mu - 2\sigma < X < \mu + 2\sigma\big) &= \int_{\mu-2\sigma}^{\mu+2\sigma} 6x(1-x)\,dx = \big(3x^2 - 2x^3\big)\Big|_{\mu-2\sigma}^{\mu+2\sigma} \\ &= \text{usual computation} = 24\mu\sigma - 24\sigma\mu^2 - 32\sigma^3 = \frac{11\sqrt{5}}{25} \approx 0.9838699101\end{aligned}$$

Meanwhile, by Chebyshev's theorem, we have

$$\mathbb{P}\big(\mu - 2\sigma < X < \mu + 2\sigma\big) = \mathbb{P}\big(|X - \mu| < 2\sigma\big) \geq 1 - \frac{1}{4} = 0.75\,,$$

which is consistent with the above probability (0.9838699101), although this does not provide a very good bound.

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Q3 (3pt) A nationwide survey of college seniors by the University of Michigan revealed that almost 70% disapprove of daily pot smoking, according to a report in *Parade*. If 12 seniors are selected at random and asked their opinion, find the probability that the number who disapprove of smoking pot daily is

(a) anywhere from 7 to 9;

(b) at most 5;

(c) not less than 8.

`Ans:` The number who disapprove of smoking pot daily is a binomial random variable with parameters $(12, 0.7)$ and let us denote it by X:

$$\mathbb{P}(X = k) = \binom{12}{k} 0.7^k 0.3^{12-k} \quad \text{for } k = 0, 1, \ldots, 12.$$

(a) $\mathbb{P}(7 \leq X \leq 9) = \mathbb{P}(X = 7) + \mathbb{P}(X = 8) + \mathbb{P}(X = 9)$, which is equal to

$$\binom{12}{7} 0.7^7 0.3^5 + \binom{12}{8} 0.7^8 0.3^4 + \binom{12}{9} 0.7^9 0.3^3 = 0.62933591328$$

(b) $\mathbb{P}(X \leq 5) = \sum_{k=0}^{5} \binom{12}{k} 0.7^k 0.3^{12-k} = 0.03860047104$

(c) $\mathbb{P}(X \geq 8) = 1 - \mathbb{P}(X \leq 5) - \mathbb{P}(X = 6) - \mathbb{P}(X = 7) = 1 - 0.03860047104 - \binom{12}{6} 0.7^6 0.3^6 - \binom{12}{7} 0.7^7 0.3^5$, which is equal to 0.48591215413.

Q4 (2pt) From a lot of 10 missiles, 4 are selected at random and fired. If the lot contains 3 defective missiles that will not fire, what is the probability that (a) all 4 will fire? (b) at most 2 will not fire?

`Ans:` (a) all 4 will fire means that the 4 good ones are chosen from the 7 functioning missiles. There are in total $\binom{7}{4} = 35$ combinations. So the probability that all 4 will fire is

$$\frac{\binom{7}{4}}{\binom{10}{4}} = \frac{35}{210} = \frac{1}{6}. \qquad \left(\text{note that } \binom{10}{4} = 210. \right)$$

(b)

$$\text{Prob}\big(\text{at most 2 missiles will not fire}\big) = 1 - \text{Prob}\big(\text{3 missiles will not fire}\big) = 1 - \frac{\binom{3}{3} \times \binom{7}{1}}{\binom{10}{4}} = 29/30.$$

Q5 (3pt) A foreign student club lists as its members 2 Canadians, 3 Japanese, 5 Italians, and 2 Germans. If a committee of 4 is selected at random, find the probability that

(a) all nationalities except Italian are represented. (1pt)

(b) only three nationalities are represented. (2pt)

`Ans:` (a) "all nationalities except Italian are represented" means one of the following combinations could be formed: (2C, 1J, 1G), (1C, 2J, 1G), (1C, 1J,2G), then the required probability is

$$\frac{\binom{2}{2} \times \binom{3}{1} \times \binom{2}{1}}{\binom{12}{4}} + \frac{\binom{2}{1} \times \binom{3}{2} \times \binom{2}{1}}{\binom{12}{4}} + \frac{\binom{2}{1} \times \binom{3}{1} \times \binom{2}{2}}{\binom{12}{4}} = \frac{6 + 12 + 6}{495} = \frac{24}{495} = 8/165$$

(b) "only three nationalities are represented" means "all nationalities except Italian are represented" or "all nationalities except Canadian are represented" or "all nationalities except Japanese are represented" or "all nationalities except Italian are represented". We know the probability that all nationalities except Italian are represented is $8/165$ from previous part. The other probabilities can be computed in the same way:

$$\begin{aligned}&\text{Prob(all nationalities except Canadian are represented)}\\&=\frac{\binom{3}{2}\times\binom{5}{1}\times\binom{2}{1}}{\binom{12}{4}}+\frac{\binom{3}{1}\times\binom{5}{2}\times\binom{2}{1}}{\binom{12}{4}}+\frac{\binom{3}{1}\times\binom{5}{1}\times\binom{2}{2}}{\binom{12}{4}}=\frac{30+60+15}{495}=7/33\,.\end{aligned}$$

and

$$\begin{aligned}&\text{Prob(all nationalities except Japanese are represented)}\\&=\frac{\binom{2}{2}\times\binom{5}{1}\times\binom{2}{1}}{\binom{12}{4}}+\frac{\binom{2}{1}\times\binom{5}{2}\times\binom{2}{1}}{\binom{12}{4}}+\frac{\binom{2}{1}\times\binom{5}{1}\times\binom{2}{2}}{\binom{12}{4}}=\frac{10+40+10}{495}=4/33\,.\end{aligned}$$

and

$$\begin{aligned}&\text{Prob(all nationalities except German are represented)}\\&=\frac{\binom{2}{2}\times\binom{3}{1}\times\binom{5}{1}}{\binom{12}{4}}+\frac{\binom{2}{1}\times\binom{3}{2}\times\binom{5}{1}}{\binom{12}{4}}+\frac{\binom{2}{1}\times\binom{3}{1}\times\binom{5}{2}}{\binom{12}{4}}=\frac{15+30+60}{495}=7/33\,.\end{aligned}$$

Hence the probability that only three nationalities are represented is equal to

$$8/165+7/33+4/33+7/33=98/165\approx 0.59393939393$$

Q6 (4pt) The refusal rate for telephone polls is known to be approximately 20%. A newspaper report indicates that 50 people were interviewed before the first refusal. (a) Comment on the validity of the report. Use a probability in your argument. (b) What is the expected number of people interviewed before a refusal?

`Ans:` Let us denote by X the number of interviewed people before the first refusal. Then $X+1$ is a random variable that follows the geometric distribution with parameter 0.2.

$$\mathbb{P}\big(X+1=k+1\big)=0.8^k 0.2 \quad \text{for} \quad k=0,1,\dots$$

just to be consistent with the definition given in the lecture.

(a) The probability that 50 people were interviewed before the first refusal is equal to $0.8^{50}\times 0.2=0.00000285449$, so the reporter is very likely a liar.

(b) $X+1$ is a geometric random variable with "success" probability 0.2, here the "success" means getting a refusal. It follows from the lecture, the expectation of such a geometric random variable has expectation

$$\mathbb{E}[X+1]=\frac{1}{0.2}=5\,.$$

So $\mathbb{E}[X]=4$, that is, the *expected number of people interviewed before a refusal is* 4.