(Due Wednesday 10/03/2018 right before class)

FALL/2018/MA526: SOLUTION TO HOMEWORK 6

Instructor: Guangqu Zheng[1]; Grader: Chessa Mccalla[2]

Total points: 20.

Q1 (4pt) Consider the probability density function of X: $f(x) = k\sqrt{x}$ for $x \in (0,1)$ and $f(x) = 0$ elsewhere. (1) First find the value of k (2pt) (2) find the CDF and use it to evaluate $\mathbb{P}(0.3 < X < 0.6)$. (2pt)

`Answer:` (1) We need

$$1 = \int_{-\infty}^{\infty} f(x)\,dx = \int_0^1 k\sqrt{x}\,dx = k\int_0^1 x^{1/2}\,dx = k \times \frac{2}{3}x^{3/2}\Big|_{x=0}^{x=1} = \frac{2}{3}k$$

implying $k = 3/2$.

(2) The CDF of X is given by

$$F_X(x) = \int_{-\infty}^{x} f(y)\,dy = \begin{cases} 0 & \text{if } x \le 0 \\ \displaystyle\int_0^x (3/2)y^{1/2}\,dy = x^{3/2} & \text{if } 0 < x < 1 \\ 1 & \text{if } x \ge 1 \end{cases}$$

Note: if students make the division like $x < 0$, $0 \le x < 1$, $x \ge 1$, they should get the point.

It follows that $\mathbb{P}(0.3 < X < 0.6) = F_X(0.6) - F_X(0.3)$, or alternatively,

$$\mathbb{P}(0.3 < X < 0.6) = \int_{0.3}^{0.6} (3/2)x^{1/2}\,dx$$

both of which will give us the right answer: $\mathbb{P}(0.3 < X < 0.6) = \dfrac{12\sqrt{15} - 3\sqrt{30}}{100}$.

Q2 If a dealer's profit, in units of \$5000, on a new automobile can be looked upon as a random variable X having the density function

$$f(x) = \begin{cases} 2(1-x) & 0 < x < 1 \\ 0 & elsewhere \end{cases}$$

(1) Find the average profit per automobile. (2pt)

(2) Find the standard deviation of profit per automobile. (2pt)

`Answer:` The expected profit per automobile is

$$(\$5000) \times \underbrace{\int_0^1 2(1-x)x\,dx}_{=\mathbb{E}[X]} = \$\frac{5000}{3} \approx \$1666.67$$

[1] gzheng90@ku.edu; Office hours: MWF 3:00-3:50; Office = 641 Snow Hall

[2] chessa_m@ku.edu

And the variance of X is equal to $\mathbb{E}[X^2] - \big(\mathbb{E}[X]\big)^2$, we know already $\mathbb{E}[X] = 1/3$. Now we compute

$$\mathbb{E}[X^2] = \int_0^1 2(1-x)x^2\,dx = 1/6$$

so that $\mathrm{Var}(X) = (1/6) - (1/3)^2 = 1/18$. So the standard deviation of X is $\sqrt{1/18} = \frac{\sqrt{2}}{6}$. **So** the standard deviation of profit per automobile is

$$(\$5000) \times \frac{\sqrt{2}}{6} = \$\frac{2500\sqrt{2}}{3} \approx \$1178.51\,.$$

Q3 Let X, Y be random variables with joint density function

$$f(x,y) = \begin{cases} 4xy & 0 < x, y < 1 \\ 0 & elsewhere \end{cases}$$

(1) Are they independent ? (2pt)

(2) Find the expected value of $\sqrt{X^2+Y^2}$. (2pt)

`Answer:` (1) Yes, they are independent. The marginal of X, the probability density function of X, is given by

$$f_X(x) = \int_0^1 f(x,y)\,dy = 4x\int_0^1 y\,dy = 2x\,, \quad \text{for } 0 < x < 1;\ f_X(x) = 0 \text{ elsewhere.}$$

In the same way, we get the probability density function $f_Y(y)$ of Y: $f_Y(y) = 2y$ for $0 < y < 1$ and $f_Y(y) = 0$ elsewhere.

So we have for any x, y, it holds that $f(x,y) = f_X(x) \times f_Y(y)$. In other words, X and Y are independent.

(2)

$$\mathbb{E}\big[\sqrt{X^2+Y^2}\big] = \int_0^1\int_0^1 \sqrt{x^2+x^2}(4xy)\,dx\,dy = \int_0^1\int_0^1 \sqrt{x^2+x^2}\,d(x^2)\,d(y^2)$$

now we make change of variable: $u = x^2$, $v = y^2$, then

$$\begin{aligned}\mathbb{E}\big[\sqrt{X^2+Y^2}\big] = \int_0^1\int_0^1 \sqrt{u+v}\,du\,dv = \int_0^1 \left(\int_0^1 \sqrt{u+v}\,du\right)\,dv &= \int_0^1 \left(\frac{2}{3}(u+v)^{3/2}\Big|_{u=0}^{u=1}\right)\,dv \\ &= \int_0^1 \left(\frac{2}{3}(1+v)^{3/2} - \frac{2}{3}v^{3/2}\right)\,dv\end{aligned}$$

It is easy to get

$$\int_0^1 \frac{2}{3}v^{3/2}dv = \frac{2}{3}\times\frac{2}{5}v^{5/2}\Big|_{v=0}^{v=1} = \frac{4}{15}\,,$$

and

$$\int_0^1 \frac{2}{3}(1+v)^{3/2}\,dv = \frac{2}{3}\times\frac{2}{5}(1+v)^{5/2}\Big|_{v=0}^{v=1} = \frac{16}{15}\sqrt{2} - \frac{4}{15}$$

hence

$$\mathbb{E}\big[\sqrt{X^2+Y^2}\big] = \frac{16}{15}\sqrt{2} - \frac{8}{15}\,.$$

Q4 (3pt) Suppose that airplane engines operate independently and fail with probability equal to 0.4. Assuming that a plane makes a safe flight if at least one-half of its engines run, determine whether a 4-engine plane or a 2- engine plane has the higher probability for a successful flight.

`Answer:` For a 4-engine plane, a successful flight means that during the flight, there are at least two engines run and its probability is equal to

$$\binom{4}{2}0.6^2 0.4^2 + \binom{4}{3}0.6^3 0.4^1 + \binom{4}{4}0.6^4 = 82.08\%$$

For a 2-engine plane, a successful flight means that during the flight, there are at least 1 engine run and its probability is equal to

$$\binom{2}{1}0.6 \times 0.4 + \binom{2}{2}0.6^2 = 84\%$$

It follows that **a 2- engine plane has the higher probability for a successful flight.**

Q5 (2pt) A safety engineer claims that only 40% of all workers wear safety helmets when they eat lunch at the workplace. Assuming that this claim is right, find the probability that 4 of 6 workers randomly chosen will be wearing their helmets while having lunch at the workplace.

`Answer:` The desired probability is

$$\binom{6}{4}0.4^4 \times 0.6^2 = 0.13824\,.$$

Q6 (3pt) Find the probability that a person flipping a coin gets the third head on the seventh flip.

`Answer:` If he/she flips a coin and gets the the third head on the seventh flip. This mean he/she gets two heads in the first 6 flips and gets the head on the seventh flip. So the probability is

$$\begin{aligned}
&\text{Prob}\big(\text{a person flipping a coin gets the third head on the seventh flip}\big)\\
&= \text{Prob}\big(\text{a person flipping a coin gets two heads on the first six flips}\big) \times \text{Prob}\big(\text{gets the head on the seventh flip}\big)\\
&= \Big(\binom{6}{4}0.5^4 \times 0.5^2\Big) \times 0.5 = 0.1171875\,.
\end{aligned}$$