FALL/2018/MA526: MID-TERM I SOLUTION

Instructor: Guangqu Zheng[1]

`Wednesday, September 26: 2:00-2:45pm`

Maximal points: 20.

Attention: `You need to provide enough supporting arguments for your response.`

Q1 (4pt) Let $x_1, \ldots, x_5$ be a sample data.

(1) Recall the definition of its sample average. (1pt)

(2) Recall the definition of its sample median. (1pt)

(3) Is sample average always bigger than sample median? (2pt) Argue with examples.

`Answer:` (1) In this question, there are only five numbers, so its sample average is given by

$$(x_1 + x_2 + x_3 + x_4 + x_5)/5$$

(2) We first relabel the numbers as y_1, y_2, y_3, y_4, y_5 such that $y_1 \le y_2 \le y_3 \le y_4 \le y_5$. If so, the median of $x_1, \ldots, x_5$ is y_3.

(3) No. The sample average may be bigger than sample median: For example $x_1 = x_2 = x_3 = x_4 = 0$, $x_5 = 5$, then the sample average is 1 while the median is zero.

The sample average may be equal to its sample median: For example $x_1 = x_2 = x_3 = x_4 = x_5 = 5$, then the sample average the median are both equal to 5.

The sample average may be smaller than its sample median: For example $x_1 = x_2 = x_3 = x_4 = 5$ and $x_5 = 0$, then the sample average is 4 & the median is 5.

To conclude, the sample average is not always bigger than the sample median.

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Q2 (4pt) Suppose you roll two fair 6-sided dice. What is the probability that the sum is at least 8 given that you rolled at least one 5?

`Answer:` We are looking for the conditional probability

$$\text{Prob}\big(\text{the sum is at least 8} \mid \text{you rolled at least one 5}\big)$$

which is equal to, by definition,

$$\frac{\text{Prob}\big(\text{the sum is at least 8 } \textbf{and} \text{ you rolled at least one 5}\big)}{\text{Prob}\big(\text{you rolled at least one 5}\big)}$$

Now back to the usual argument: when we roll two fair dice, we get 36 possible outcomes, each of which has equal probability 1/36:

$$\Omega := \big\{(1,1),(1,2),\dots,(1,6),(2,1),(2,2),\dots,(2,6),\dots,(6,6)\big\}$$

The event {the sum is at least 8 **and** you rolled at least one 5} contains the following outcomes:

$$\big\{(3,5),(4,5),(5,5),(6,5),(5,3),(5,4),(5,6)\big\}$$

this event has probability 7/36.

The event {you rolled at least one 5} contains the following outcomes:

$$\big\{(1,5),(2,5)(3,5),(4,5),(5,5),(6,5),(5,1),(5,2),(5,3),(5,4),(5,6)\big\}$$

this event has probability 11/36.

So the required condition probability is 7/11.

Q3-(1) (3pt) Determine the value C so that the following function can serve as a probability density function of the random variable X:

$f(x) = Cx^2$ for $0 \le x \le 1$; $f(x) = C|x|$ for $-2 \le x \le -1$; $f(x) = 0$ elsewhere.

`Answer:` First it is clear that we require $C > 0$. And we also need

$$1 = \int_{-\infty}^{\infty} f(x)dx = \int_0^1 Cx^2\,dx + \int_{-2}^{-1} C|x|\,dx$$

Note that

$$\int_0^1 Cx^2\,dx = C\int_0^1 x^2\,dx = C\Big(\frac{x^3}{3}\Big)\Big|_{x=0}^{x=1} = C/3$$

while (as $|x| = -x$ for $x < 0$)

$$\int_{-2}^{-1} C|x|\,dx = -C\int_{-2}^{-1} x\,dx = -C\Big(\frac{x^2}{2}\Big)\Big|_{x=-2}^{x=-1} = \frac{3}{2}C$$

Hence $1 = \dfrac{C}{3} + \dfrac{3}{2}C = \dfrac{11}{6}C$ implying $C = 6/11$.

Q3-(2)(4pt) Two fair dice are rolled. Consider the events $A = \{$ sum of two dice equals 3 $\}$,

$B = \{$ sum of two dice equals 7 $\}$ and $C = \{$ at least one of the dice shows a 1$\}$.

(a) What is $\mathbb{P}(A|C)$? (1pt)

(b) What is $\mathbb{P}(B|C)$? (1pt)

(c) Are event A and event C independent? What about B and C? (2pt)

`Answer:`

(a) $\mathbb{P}(A|C) = \mathbb{P}(A \cap C)/\mathbb{P}(C)$ and the event $A \cap C$ means sum of two dice is three **and** at least one of the dice shows 1, that it is equivalent to say that (1, 2) or (2,1) are rolled. So $\mathbb{P}(A \cap C) = 2/36 = 1/18$. (`Note that in this question, we have` $A \subset C$)

And the event C means that one of $(1,1),(1,2),\ldots,(1,6),(2,1),(3,1),(4,1),(5,1),(6,1)$ is rolled, so $\mathbb{P}(C) = 11/36$.

Thus, $\mathbb{P}(A|C) = \mathbb{P}(A \cap C)/\mathbb{P}(C) = 2/11$.

(b) $\mathbb{P}(B|C) = \mathbb{P}(B \cap C)/\mathbb{P}(C)$ and the event $B \cap C$ means sum of two dice is 7 **and** at least one of the dice shows 1, that it is equivalent to say that (1, 6) or (6,1) are rolled. So $\mathbb{P}(B \cap C) = 2/36 = 1/18$.

And we know from part (a) that $\mathbb{P}(C) = 11/36$. Thus, $\mathbb{P}(B|C) = \mathbb{P}(B \cap C)/\mathbb{P}(C) = 2/11$.

(c) The event A and event C are **not independent**, the event B and event C are **not independent** either: We know from part (a) that $\mathbb{P}(A|C) = \mathbb{P}(A)/\mathbb{P}(C) \neq \mathbb{P}(A)$, as $A \subset C$; and we know from part (b) that $\mathbb{P}(B|C) = 2/11$ while $\mathbb{P}(B) = 6/36$.

The result $\mathbb{P}(B) = 6/36$ comes from the fact that the event B means that one of $(1,6),(6,1),(2,5),(5,2),(3,4),(4,3)$ is rolled.

Q4: Next page

Bonus question (3 pt) Let X be a random variable with finite second moment and write $\mathbb{E}$ for the expectation. Then what is the minimal value for the function $\phi(x) := \mathbb{E}\big([X-x]^2\big)$?

`Answer:`

We know $(y-x)^2 = y^2 - 2xy + x^2$, so $[X-x]^2 = X^2 - 2xX + x^2$. Therefore,

$$\phi(x) := \mathbb{E}\big([X-x]^2\big) = \mathbb{E}\big(X^2 - 2xX + x^2\big) = \mathbb{E}[X^2] - 2\mathbb{E}[X]x + x^2$$

This is a function in the variable x, while $\mathbb{E}[X^2]$ and $2\mathbb{E}[X]$ are just numbers. It is enough to find x_0 such that $\phi'(x_0) = 0$, as we know $\phi(x)$ has a unique minimum. Let us compute the derivative of $\phi(x)$ now:

$$\frac{d}{dx}\phi(x) = 2x - 2\mathbb{E}[X]$$

so that we should take $x_0 = \mathbb{E}[X]$ and $\phi(x_0) = \mathbb{E}\big([X - \mathbb{E}(X)]^2\big) = \mathrm{Var}(X)$ is the minimal value of the function ϕ.

Q4 (5pt) Let Ω be a sample space equipped with a probability $\mathbb{P}$. Let A, B be two events on Ω such that A, B are independent and $\mathbb{P}(A) = \mathbb{P}(B) = 1/2$. Now we define the random variable $X(\omega) = \mathbb{I}_A(\omega) + \mathbb{I}_B(\omega)$, where the indicator function $\mathbb{I}_A$ is defined by

$$\mathbb{I}_A(\omega) = 1 \quad \text{if } \omega \in A \quad \textit{and} \quad \mathbb{I}_A(\omega) = 0 \quad \text{if } \omega \text{ is not in } A.$$

(1) Find $\mathbb{P}(A \cap B)$ (1pt) and find $\mathbb{P}(A^c \cap B^c)$ (1pt)
(2) Write down the probability mass function of X. (1pt)
(3) Write down the cumulative distributional function of X. (1pt)
(4) Compute the expectation of X. (1pt)

`Answer:`

(1) Since A, B are independent, we have $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B) = 1/4$. By de-Morgan's law $(A \cup B)^c = A^c \cap B^c$, then

$$\mathbb{P}(A^c \cap B^c) = 1 - \mathbb{P}(A \cup B) = 1 - \Big[\mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)\Big]$$

where the last equality follows from the additive rules. So we have $\mathbb{P}(A^c \cap B^c) = 1 - \Big[1/2 + 1/2 - 1/4\Big] = 1/4$.

(2) It is clear to see the random variable X defined above is discrete with possible values in $\{0, 1, 2\}$. And we have

$$X(\omega) = 0 \quad \text{if and only if} \quad \mathbb{I}_A(\omega) = \mathbb{I}_B(\omega) = 0 \quad \text{if and only if} \quad \omega \in A^c \cap B^c$$

In other words, $\{X = 0\} = A^c \cap B^c$, therefore $\mathbb{P}(X = 0) = 1/4$. And similarly, we have

$$X(\omega) = 2 \quad \text{if and only if} \quad \mathbb{I}_A(\omega) = \mathbb{I}_B(\omega) = 1 \quad \text{if and only if} \quad \omega \in A \cap B$$

In other words, $\{X = 2\} = A \cap B$, therefore $\mathbb{P}(X = 0) = 1/4$. Hence

$$\mathbb{P}(X = 1) = 1 - \mathbb{P}(X = 0) - \mathbb{P}(X = 2) = 1/2$$

implying that the probability mass function f_X of the random variable X is given by

$$f_X(x) = \begin{cases} 1/4 & \text{if } x = 0 \\ 1/2 & \text{if } x = 1 \\ 1/4 & \text{if } x = 2 \\ 0 & \text{elsewhere.} \end{cases}$$

(3) The corresponding cumulative distribution function F_X is given by

$$F_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1/4 & \text{if } 0 \le x < 1 \\ 3/4 & \text{if } 1 \le x < 2 \\ 1 & \text{if } x \ge 2. \end{cases}$$

(4) By definition,

$$\mathbb{E}\big[X\big] = \sum_x x f_X(x) = 0 \times \frac{1}{4} + 1 \times \frac{1}{2} + 2 \times \frac{1}{4} = 1\,.$$

Alternatively, we have $\mathbb{E}\big[X\big] = \mathbb{E}\big[\mathbb{I}_A + \mathbb{I}_B\big] = \mathbb{E}\big[\mathbb{I}_A\big] + \mathbb{E}\big[\mathbb{I}_B\big] = \mathbb{P}(A) + \mathbb{P}(B) = 1$.