(Due Wednesday 09/19/2018 right before class)
&
(Your homework shall be stapled if it contains multiple pages.)

FALL/2018/MA526: SOLUTION TO HOMEWORK 4

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Total points: 20.

Q1 Students shall show enough details for obtaining their answers.

(1) Find the value of C such that f defined below is a probability density function: (2 points)

$$f(x) = \begin{cases} C \cdot x^{-3} & \text{if } x \geq 2 \\ 0 & \text{if } x < 2. \end{cases}$$

Ans: C should satisfy

$$\int_2^\infty C\, x^{-3}\, dx = 1.$$ We know from Calculus that

$$\int_2^\infty \frac{1}{x^3}\, dx = \int_2^\infty d\left(\frac{x^{-2}}{-2}\right) = \frac{x^{-2}}{-2}\Big|_2^\infty = \frac{1}{8}$$

$$\Downarrow$$

$$\boxed{C = 8}$$

(2) Find the value of C such that g defined below is a probability density function: (3 points)

$$g(x) = \begin{cases} C \cdot x^{-3} & \text{if } x \geq 2 \\ C \cdot x^{-2} & \text{if } 1 \leq x \leq 2. \\ 0 & \text{if } x < 1. \end{cases}$$

Ans: C should satisfy

$$\underbrace{\int_2^\infty C\, x^{-3}\, dx}_{\frac{1}{8}C \text{ by (1)}} + \int_1^2 C\, x^{-2}\, dx = 1$$

$$\int_1^2 \frac{1}{x^2}\, dx = \left(-\frac{1}{x}\right)\Big|_{x=1}^{x=2} = 1 - \frac{1}{2} = \frac{1}{2}$$

Hence $\frac{1}{8}C + \frac{1}{2}C = 1 \Rightarrow \frac{5}{8}C = 1 \Rightarrow C = \frac{8}{5}$

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(3) Find the values of A and B such that h defined below is a probability density function: (3 points)

$$h(x) = \begin{cases} Ae^{-x} & \text{if } x \geq 1 \\ Bx & \text{if } -1 \leq x \leq 1. \\ \frac{1}{2}e^{x} & \text{if } x \leq -1. \end{cases}$$

Ans: $h(x) \geq 0$ for any $x \in \mathbb{R}$, so B has to be zero.

while $\int_{-\infty}^{\infty} h(x)\,dx = 1 \Rightarrow \int_{1}^{\infty} A e^{-x}\,dx + \int_{-\infty}^{-1} \frac{1}{2} e^{x}\,dx = 1$

AND $\int_{1}^{\infty} e^{-x}\,dx = e^{-1}$ & $\int_{-\infty}^{-1} e^{x}\,dx = e^{-1}$, so $e^{-1}A + \frac{1}{2}e^{-1} = 1 \Rightarrow \boxed{\cancel{A = \frac{1-e^{-1}}{e^{-1}} = e-1}}$

$A = e - \frac{1}{2}$

(4) Solve **exercise 3.12 page 92** in the *textbook* (see the following screenshot) (4 points)

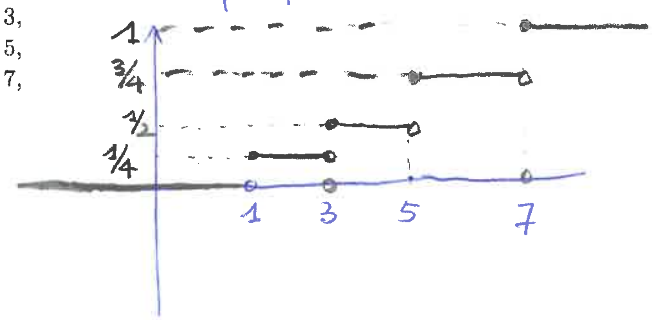
3.12 An investment firm offers its customers municipal bonds that mature after varying numbers of years. Given that the cumulative distribution function of T, the number of years to maturity for a randomly selected bond, is

$$F(t) = \begin{cases} 0, & t < 1, \\ \frac{1}{4}, & 1 \leq t < 3, \\ \frac{1}{2}, & 3 \leq t < 5, \\ \frac{3}{4}, & 5 \leq t < 7, \\ 1, & t \geq 7, \end{cases}$$

find

(a) $P(T = 5)$;
(b) $P(T > 3)$;
(c) $P(1.4 < T < 6)$;
(d) $P(T \leq 5 \mid T \geq 2)$.

Graph of CDF F(·)



Ans: Note first that T is a discrete random variable with finite many values & T is integer-valued

(a) We can read from graph of F that $\mathbb{P}(T=5) = \frac{3}{4} - \frac{1}{2} = \frac{1}{4}$

(b) $\mathbb{P}(T>3) \underset{\text{see graph}}{=} \mathbb{P}(T=5) + \mathbb{P}(T=7)$ $\quad \mathbb{P}(T=7) = \frac{1}{4}$

or $\mathbb{P}(T>3) = 1 - \mathbb{P}(T \leq 3) = 1 - F(3) = 1 - \frac{1}{2} = \boxed{\frac{1}{2}}$

(c) $\mathbb{P}(1.4 < T < 6) = \mathbb{P}(T=3) + \mathbb{P}(T=5)$ $\Big\} \Rightarrow \mathbb{P}(1.4<T<6) = \frac{1}{2}$

$\mathbb{P}(T=3) \overset{\text{graph}}{=} \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$

(d) $\mathbb{P}(T \leq 5 \mid T \geq 2) = \mathbb{P}(2 \leq T \leq 5)/\mathbb{P}(T \geq 2) = \frac{\mathbb{P}(T=3) + \mathbb{P}(T=5)}{1 - \mathbb{P}(T=1)} = \frac{\frac{1}{2}}{\frac{3}{4}} = \boxed{\frac{2}{3}}$

Q2 (4 points) Let W be a random variable giving the number of heads minus the number of tails in three tosses of a coin.

1) List the elements of the sample space Ω for the three tosses of the coin. (1 point)

2) Assume that the probability of getting a head is equal to 1/2. Find out the probability mass function f associated to W. (2 points)

3) Find out the cumulative distributional function associated to the pmf f given in part-2. (1 point)

Ans: ① $\Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$

H means head
T means tail.

HHT means in three tosses, we get H, H, T in order.

② $W(HHH) = 3$ $W(HHT) = 2-1 = 1$ $W(HTH) = 2-1 = 1$ $W(HTT) = 1-2 = -1$

$W(THH) = 2-1 = 1$ $W(THT) = 1-2 = -1$ $W(TTH) = 1-2 = -1$ $W(TTT) = -3$

So W can ONLY take values $-1, -3, 1, 3$ $\Rightarrow$ $f(x) = 0$ for $x \notin \{-1, 1, 3, -3\}$

and $f(-1) = \mathbb{P}(W=-1) = \mathbb{P}(\{HTT, THT, TTH\}) = \boxed{\frac{3}{8}}$ (each outcome has equal proba)

$f(1) = \mathbb{P}(W=1) = \mathbb{P}(\{HHT, HTH, THH\}) = \boxed{\frac{3}{8}}$ $f(3) = \mathbb{P}(W=3) = \mathbb{P}(\{HHH\}) = \boxed{\frac{1}{8}}$

$f(-3) = \mathbb{P}(W=-3) = \mathbb{P}(\{TTT\}) = \boxed{\frac{1}{8}}$

Q3 (4 points) Let A, B, C be three events on some sample space Ω such that A and B are independent while $(A \cup B) \cap C = \emptyset$. Assume that $\mathbb{P}(C) = 0.1$, $\mathbb{P}(A) = 0.5$ and $\mathbb{P}(B) = 0.4$.

We define now

$$X = \mathbb{I}_A + \mathbb{I}_B + \mathbb{I}_C .$$

(a) Prove A and C are not independent. (1 point) (b) Find the CDF of X. (3 points)

Recall $\mathbb{I}_A$ is the indicator function over A, defined as follows:

$$\mathbb{I}_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

(a) Proof $\mathbb{P}(A \cap C) = \mathbb{P}(\phi) = 0$ as $(A \cup B) \cap C = \phi$.

But $\mathbb{P}(A) > 0$, $\mathbb{P}(C) > 0 \Rightarrow \mathbb{P}(A)\,\mathbb{P}(C) > 0$

$\Rightarrow$ A and C are not independent

(b) let us first find the pmf $f(x)$ of X.

Note that X can only take three values i.e. $0, 1, 2, \boxed{3}$ impossible

• If $\mathbb{I}_C = 1$, because $(A \cup B) \cap C = \phi$, $\mathbb{I}_A = 0 = \mathbb{I}_B$

Similarly if $\mathbb{I}_A = 1$ or $\mathbb{I}_B = 1$, we have $\mathbb{I}_C = 0$

So $\mathbb{P}(X=2) = \mathbb{P}(A \cap B) \underset{\text{indep}}{=} \mathbb{P}(A)\,\mathbb{P}(B) = 0.2$, & $\mathbb{P}(X=3) = 0$

See Next page

$$\mathbb{P}(X=1) \overset{\text{three cases}}{=} \mathbb{P}(I_C=1, I_A=I_B=0)$$
$$+\ \mathbb{P}(I_A=1, I_B=I_C=0)$$
$$+\ \mathbb{P}(I_B=1, I_A=I_C=0)$$

As discussed, when $I_C=1$, we have $I_A=I_B=0$

and

when I_A or $I_B=1$, we have $I_C=0$

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we get

$$\mathbb{P}(X=1) = \mathbb{P}(I_C=1) + \mathbb{P}(I_A=1, I_B=0)$$
$$+\ \mathbb{P}(I_B=1, I_A=0)$$

$$= \underbrace{\mathbb{P}(C)}_{0.1} + \mathbb{P}(A\cap B^c) + \mathbb{P}(B\cap A^c)$$

$$\mathbb{P}(A\cap B^c) = \mathbb{P}(A) - \mathbb{P}(A\cap B) = \mathbb{P}(A) - \mathbb{P}(A)\mathbb{P}(B)$$
$$= 0.5 - 0.5\times 0.4$$
$$= 0.3$$

$$\mathbb{P}(B\cap A^c) = \mathbb{P}(B) - \mathbb{P}(A\cap B)$$
$$= \mathbb{P}(B) - \mathbb{P}(A)\mathbb{P}(B) = 0.4 - 0.5\times 0.4 = 0.2$$

So $\mathbb{P}(X=1) = 0.1 + 0.3 + 0.2 = 0.6$

and $\mathbb{P}(X=0) = 1 - 0.6 - 0.2 = 0.2$

CDF

$$F(x) = \begin{cases} 0 & x<0 \\ 0.2 & 0\le x<1 \\ 0.8 & 1\le x<2 \\ 1 & x\ge 2 \end{cases}$$