(Due Wednesday 09/12/2018 right before class)

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(Your homework shall be stapled if it contains multiple pages.)

FALL/2018/MA526: SOLUTIONS TO HOMEWORK 3

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Total points: 20.

Q1 (6 points) A pair of fair dice are tossed. Find the probability of getting (1) a total of 8 (2) at most a total 5. Students should specify the sample space Ω, the probability $\mathbb{P}$ on Ω and the events that they are asked to consider.

`Answer:` Write (i,j) to mean that the first dice gives us i and the second gives us j for any given $i,j\in\{1,\dots,6\}$. Then

$$\Omega=\Big\{\,(1,1),(1,2),\dots,(1,6),(2,1),(2,2),\dots,(2,6),\dots,(3,1),\dots,(6,6)\,\Big\}$$

is the sample space.

Ω has 36 elements (that is, there are 36 possible outcomes from throwing two fair dice) and each outcome has the same probability, that is,

$$\mathbb{P}(\{(i,j)\})=1/36$$

for any $i,j\in\{1,2,3,4,5,6\}$.

Event described in (1) is $E_1:=\big\{(i,j)\in\Omega\,:\,i+j=8\big\}$, which is the same as

$$\Big\{\,(2,6),(6,2),(3,5),(5,3),(4,4)\,\Big\}$$

Event described in (2) is $E_2:=\big\{(i,j)\in\Omega\,:\,i+j\le 5\big\}$, which is the same as

$$\Big\{\,(1,1),(1,2),(2,1),(1,3),(3,1),(1,4),(4,1),(2,2),(2,3),(3,2)\,\Big\}$$

It follows that $\mathbb{P}(E_1)=5/36$ and $\mathbb{P}(E_2)=10/36=5/18$.

Q2 (3 points) Bag 1 contains 5 white balls and 4 black balls while Bag 2 contains 3 white balls and 5 black balls. Assume that your eyes have been ALWAYS covered so that you can not distinguish different balls in any of these two bags. One ball was drawn from bag 1 and placed into bag 2. Now you draw a ball from bag 2, then, what is the probability that you get a black ball?

`Answer:` This is an exercise about the *formula of total probability.* Note that the first drawing only gives us two events, that is, a black ball was drawn or a white ball was drawn, denoted by D_B and D_W respectively. Let Ω be the sample space under consideration, then $D_B\cup D_W=\Omega$ and they are disjoint. So D_W, D_B form a partition of the sample space. It follows then:

$$\texttt{Prob}\big(\text{you get a black ball}\big)=\texttt{Prob}\big(\text{you get a black ball and }D_B\big)+\texttt{Prob}\big(\text{you get a black ball and }D_W\big)$$
$$=\texttt{Prob}\big(\text{you get a black ball }|D_B\big)\texttt{Prob}(D_B)+\texttt{Prob}\big(\text{you get a black ball }|D_W\big)\texttt{Prob}(D_W)$$

In bag 1, there are 5 white balls and 4 black balls, so $\texttt{Prob}(D_B)=4/9$ and $\texttt{Prob}(D_W)=5/9$. If D_B happened, there are 3 white balls and 6 black balls so that $\texttt{Prob}\big(\text{you get a black ball }|D_B\big)=6/9=2/3$; while if D_W happened, there are 4 white balls and 5 black balls so that $\texttt{Prob}\big(\text{you get a black ball }|D_W\big)=5/9$

therefore the answer is $\frac{4}{9}\times\frac{2}{3}+\frac{5}{9}\times\frac{5}{9}=49/81$.

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Q3 (8 points) Pollution of the rivers in the United States has been a problem for many years. Consider the following events:

A: the river is polluted,

B: a sample of water tested detects pollution,

C : fishing is permitted.

Assume $\mathbb{P}(A) = 0.3$, $\mathbb{P}(B|A) = 0.75$, $\mathbb{P}(B|A^c) = 0.2$, $\mathbb{P}(C|A \cap B) = 0.2$, $\mathbb{P}(C|A^c \cap B) = 0.15$, $\mathbb{P}(C|A \cap B^c) = 0.8$ and $\mathbb{P}(C|A^c \cap B^c) = 0.9$, then find (1) $\mathbb{P}(A \cap B \cap C)$ (2) $\mathbb{P}(B^c \cap C)$ (3) $\mathbb{P}(C)$ (4) Find the probability that the river is polluted, given that fishing is permitted and the sample tested did not detect pollution.

`Hint:` Another de-Morgan's law $(A \cap B)^c = A^c \cup B^c$.

`Ans:` (1) $\mathbb{P}(A) = 0.3$ and $\mathbb{P}(B|A) = 0.75$ imply together that $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B|A) = 9/40$. Since $\mathbb{P}(C|A \cap B) = 0.2$,

$$\mathbb{P}(A \cap B \cap C) = \mathbb{P}(C|A \cap B) \times \mathbb{P}(A \cap B) = \frac{9}{200} \quad \text{or} \quad 0.045\,.$$

(2) Let us find out the probability of B: by the formula of total probabilities

$$\mathbb{P}(B) = \mathbb{P}(B|A)\mathbb{P}(A) + \mathbb{P}(B|A^c)\mathbb{P}(A^c) = \frac{3}{4} \times \frac{3}{10} + \frac{1}{5} \times \frac{7}{10} = \frac{73}{200}$$

and this implies

$$\mathbb{P}(A^c \cap B^c) = 1 - \mathbb{P}\Big(\big(A^c \cap B^c\big)^c\Big) = 1 - \mathbb{P}\Big(A \cup B\Big) = 1 - \big[\mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)\big]$$
$$= 1 - \left(\frac{3}{10} + \frac{73}{200} - \frac{9}{40}\right) = 0.56$$

and this further implies

$$\mathbb{P}(C \cap A^c \cap B^c) = \mathbb{P}(C|A^c \cap B^c)\mathbb{P}(A^c \cap B^c) = 0.9 \times 0.56 = 0.504\,.$$

Recall we want the value of $\mathbb{P}(B^c \cap C)$, which is equal to $\mathbb{P}(C \cap A^c \cap B^c) + \mathbb{P}(C \cap A \cap B^c)$. Now we need to find $\mathbb{P}(C \cap A \cap B^c)$:

$$\mathbb{P}(C \cap A \cap B^c) = \mathbb{P}(C|A \cap B^c)\mathbb{P}(A \cap B^c) = 0.8 \times \mathbb{P}(A \cap B^c) = 0.8 \times \Big[\mathbb{P}(A) - \mathbb{P}(A \cap B)\Big] = 0.8 \times \Big[0.3 - 9/40\Big] = 0.06$$

therefore,

$$\mathbb{P}(B^c \cap C) = \mathbb{P}(C \cap A^c \cap B^c) + \mathbb{P}(C \cap A \cap B^c) = 0.504 + 0.06 = 0.564$$

(3) We first have from point (2) that $\mathbb{P}(A \cap C) = \mathbb{P}(C \cap A \cap B^c) + \mathbb{P}(C \cap A \cap B) = 0.06 + 0.045 = 0.105$ and note that $A^c \cap B$, $A^c \cap B^c$ and A form a partition of the sample space so by the formula of total probabilities, we get

$$\begin{aligned}\mathbb{P}(C) = \mathbb{P}(C \cap A^c \cap B) + \mathbb{P}(C \cap A^c \cap B^c) + \mathbb{P}(C \cap A) &= \mathbb{P}(C|A^c \cap B)\mathbb{P}(A^c \cap B) + \mathbb{P}(C \cap A^c \cap B^c) + \mathbb{P}(C \cap A) \\ &= 0.15 \times \mathbb{P}(A^c \cap B) + 0.504 + 0.105 \\ &= 0.15 \times \big[\mathbb{P}(B) - \mathbb{P}(A \cap B)\big] + 0.504 + 0.105 \\ &= 0.15 \times \Big[\frac{73}{200} - \frac{9}{40}\Big] + 0.504 + 0.105 = 0.63\end{aligned}$$

(4) The probability asked for here is $\mathbb{P}(A|C \cap B^c) = \dfrac{\mathbb{P}(A \cap C \cap B^c)}{\mathbb{P}(C \cap B^c)} = \dfrac{0.06}{0.564} \approx 0.10638297872$

Q4 (3 points) Assume that in a family the birth of a boy and a girl is equally likely and that the family has $n \geq 2$ children. Are the events A, B defined below independent?

$A = \{$There is at least one boy and at least one girl in the family$\}$

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$B = \{$There is at most one girl in the family$\}$.

`Answer:` A, B are not independent. Let us consider the simple case where $n = 2$. **In this case**, $A \cap B$ stands for the event that there is one girl and one boy in the family so $\mathbb{P}(A \cap B) = 1/2$: as we only have two combinations (boy, girl) and (girl boy) each with probability $1/4$.

Let us look at A^c and B^c: A^c stands for the event that there are two boys in the family or two girls in the family. so $\mathbb{P}(A^c) = 1/2$ and $\mathbb{P}(A) = 1/2$

similarly, B^c stands for the event that there are two girls in the family, so $\mathbb{P}(B^c) = 1/4$ and $\mathbb{P}(B) = 3/4$.

So in the case we have

$$\mathbb{P}(A \cap B) = 1/2 \neq 3/8 = \mathbb{P}(A)\mathbb{P}(B)$$

that is, A and B are not independent.