(Due Wednesday 09/05/2018 right before class)

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(Your homework shall be stapled if it contains multiple pages.)

FALL/2018/MA526: SOLUTIONS FOR HOMEWORK 2

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Total points: 20. This homework focus on (i) counting (ii) basic set operations (iii) the concept of probability and its additive rules.

Q1 (3 points) List the elements of each of the following sample spaces:

(a) the set of integers between 1 and 10.89999.
(b) the set $S = \{x : 2x - 2 < 10 \text{ and } x + 1 > 4\}$.
(c) the set of outcomes when a coin is tossed until a tail or three heads appear.

`Answer:` (a) $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.

(b) $2x - 2 < 10$ gives us $x < 6$ while $x + 1 > 4$ gives us $x > 3$, so combining these two conditions together, we get $S = (3, 6)$, the open interval from 3 to 6.

(c) Note that when you toss a coin, you get head or tail, denoted by H, T respectively. According to our rule, the coin flipping will be terminated once we get T or HHH. We can list all the possible outcomes as follows:

- if the first coin tossing gives us T, then the experiment is terminated.
- if the first coin tossing gives us H, we can continue and we may get HT, HH as possible outcomes of two coin tossing. Note that if we get HT, the experiment is terminated.
- if the first two coin tossing gives us HH, we can continue and we may get HHH and HHT as possible outcomes of three coin tossing. In this case, we have to terminate our experiment according to our rule.

Hence the set of outcomes according to our rule is given by $\{T, HT, HHH, HHT\}$. You can verify that the probabilities of outcomes in the sample space sum up to one.

Q2 (3 points) Let the sample space Ω be $\{1, 2, \dots\}$ and let $p_i = 2^{-i}$ for each integer $i \geq 1$. Verify that $\{p_i, i \geq 1\}$ defines a sequence of probabilities on Ω.

`Answer:` It is enough to verify that $\sum_{i=1}^{\infty} p_i = 1$, which is an exercise about geometric series. If you know $\sum_{i=1}^{\infty} p_i$ is finite, say equals x, then you can find the value of x as follows:

$$x = \sum_{i=1}^{\infty} \frac{1}{2^i} \quad \text{and} \quad 2x = 2\left(\frac{1}{2} + \frac{1}{4} + \dots\right) = 1 + \frac{1}{2} + \frac{1}{4} + \dots = 1 + x$$

so that $2x = 1 + x$, implying $x = 1$.

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Q3 (5 points) Recall the De-Morgan's law: Given three sets $A, B, C \subset \Omega$, one has

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \quad \text{and} \quad A \cup (B \cap C) = (A \cup B) \cap (A \cup C)\,. \tag{0.1}$$

Assume $\mathbb{P}$ is a probability on the sample space Ω, then use the above De-Morgan's law to show

$$\mathbb{P}\big(A \cup B \cup C\big) = \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) - \mathbb{P}(A \cap B) - \mathbb{P}(A \cap C) - \mathbb{P}(B \cap C) + \mathbb{P}(A \cap B \cap C)\,. \tag{0.2}$$

Note for students: `You are not forced to use De-Morgan's law, it is just a hint. You may be able to find other ways to prove it.`

`Answer:` You heard the de-Morgan's law several times, have you ever tried to prove it? Here I will just prove the left equality in (0.1), then you may want to prove the right one. Note this is not a part of the homework assignment, just FYI.

`Proof of` (0.1)`/2:` Suppose $A \cap \big(B \cup C\big) = \emptyset$, this means that A has no common element with B or C, or equivalently, $A \cap B = \emptyset = A \cap C$, so we have the desired equality in this case.

Now we assume that $A \cap \big(B \cup C\big) \neq \emptyset$, this means A has at least one element that is also in $B \cup C$, so that $(A \cap B) \cup (A \cap C)$ is neither empty. Under this assumption, we have the following equivalence:

$$\begin{aligned} x \in A \cap \big(B \cup C\big) &\Longleftrightarrow x \in A \quad \text{and} \quad x \in B \cup C \Longleftrightarrow \underline{x \in A} \quad \text{and} \quad \underline{x \in B \quad \text{or} \quad C} \\ &\Longleftrightarrow \underline{x \in A \quad \text{and} \quad x \in B} \quad \text{or} \quad \underline{x \in A \quad \text{and} \quad C} \\ &\Longleftrightarrow x \in A \cap B \quad \text{or} \quad x \in A \cap C \Longleftrightarrow x \in \big(A \cap B\big) \cup \big(A \cap C\big) \end{aligned}$$

this proves $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. `End of the proof.`

`Proof of` (0.2)`:` For the convenience, we write $D = B \cup C$ for now: we have then

$$\mathbb{P}\big(A \cup B \cup C\big) = \mathbb{P}\big(A \cup D\big) = \mathbb{P}(A) + \mathbb{P}(D) - \mathbb{P}(A \cap D) = \mathbb{P}(A) + \mathbb{P}(B \cup C) - \mathbb{P}\big[A \cap (B \cup C)\big] \tag{0.3}$$

and

$$\mathbb{P}(B \cup C) = \mathbb{P}(B) + \mathbb{P}(C) - \mathbb{P}(B \cap C) \tag{0.4}$$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \tag{0.5}$$

from (0.5), we get

$$\begin{aligned} \mathbb{P}\big[A \cap (B \cup C)\big] &= \mathbb{P}\big[(A \cap B) \cup (A \cap C)\big] = \mathbb{P}\big[A \cap B\big] + \mathbb{P}\big[A \cap C\big] - \mathbb{P}\big[(A \cap B) \cap (A \cap C)\big] \\ &= \mathbb{P}\big[A \cap B\big] + \mathbb{P}\big[A \cap C\big] - \mathbb{P}\big[A \cap B \cap C\big] \end{aligned} \tag{0.6}$$

Note that in these chain of equalities, we used several times the "additive rule" $\mathbb{P}\big(A \cup B\big) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$!!!!!

To conclude, it is enough to plug (0.4) and (0.6) into (0.3). `End of the proof.`

Q4 ($3 + 2$ points) Let Ω be a sample space, equipped with a probability $\mathbb{P}$. Suppose A, B are two events with strictly positive probability.

(1) If we define $Q(C) = \mathbb{P}(C|A)$ for any $C \subset \Omega$. Show that Q is also a probability on Ω.

(2) If $\mathbb{P}(B|A) = \mathbb{P}(B)$, prove $\mathbb{P}(A|B) = \mathbb{P}(A)$.

Proof. (1) To show Q is a probability we only need to verify three points: (i) for any event $C \subset \Omega$, $Q(C) = \mathbb{P}(C|A) = \mathbb{P}(A \cap C)/\mathbb{P}(A) \in [0, 1]$ (*why?*) `students shall indicate the fact that` $A \cap C \subset A$ `so that` $\mathbb{P}(A \cap C) \leq \mathbb{P}(A)$.

(ii) let $C_1, C_2, \ldots$ be a sequence of **mutually disjoint** events, then

$$Q\left(\bigcup_{i=1}^{\infty} C_i\right) = \frac{1}{\mathbb{P}(A)}\mathbb{P}\left(A \cap \bigcup_{i=1}^{\infty} C_i\right) = \frac{1}{\mathbb{P}(A)}\mathbb{P}\left(\bigcup_{i=1}^{\infty} (A \cap C_i)\right)$$

where in the last equality we used a generalized version of de-Morgan's law. As C_i are mutually disjoint and $A \cap C_i \subset C_i$, we have $A \cap C_i$, $i \geq 1$, are also mutually disjoint so that the additive rule for the probability $\mathbb{P}$ implies

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty}(A \cap C_i)\right) = \sum_{i=1}^{\infty}\mathbb{P}\big(A \cap C_i\big)$$

then we have

$$Q\left(\bigcup_{i=1}^{\infty} C_i\right) = \frac{1}{\mathbb{P}(A)}\sum_{i=1}^{\infty}\mathbb{P}\big(A \cap C_i\big) = \sum_{i=1}^{\infty}\frac{\mathbb{P}\big(A \cap C_i\big)}{\mathbb{P}(A)} = \sum_{i=1}^{\infty} Q(C_i)$$

(iii) It is clear that (but `students should point it out`) $Q(\emptyset) = 0$ and $Q(\Omega) = 1$, as $\emptyset \cap A = \emptyset$ and $\Omega \cap A = A$.

These three points show us that Q is indeed a probability on Ω.

(2) By definition, $\mathbb{P}(B|A) = \mathbb{P}(B)$ implies $\frac{\mathbb{P}(B \cap A)}{\mathbb{P}(A)} = \mathbb{P}(B)$, so that $\mathbb{P}(B \cap A) = \mathbb{P}(A)\mathbb{P}(B)$. Therefore,

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \mathbb{P}(A)\,.$$

`If the student argues like "the condition` $\mathbb{P}(B|A) = \mathbb{P}(B)$ `implies the independence of these two events while independence is commutative, then the desired result follows immediately", the grader will give the full point for this question (2).`

□

Q5 (2+2 points) (a) How many distinct permutations can be made from the letters of the word **INFINITY**?

(b) How many ways are there that no two students will have the same birth date in a class of size 45?

Note for students: `we only consider the month and day as the birth date, for example 01/26/1990 and 01/26/1909 are identified as the same birth date.`

`Answer:`

(a) there are three letters **I**, two letters **N** and one letter **F**, one letter **T** and one letter **Y** in the word **INFINITY**. As we only want the number of distinct permutations, it is equivalent to look at the total number of distinct permutations can be made from (1, 1, 1, 2, 2, 3, 4,5):

In total there are 8 numbers with repetitions. Think about they are placed in 8 different boxes, now if you permute the boxes, you will get 8! different ways of rearranging the boxes. Now you look inside the boxes and ignore the boxes outside, then you could not tell if the 1's are permuted or not because of repetitions. So you need to divide 8! by (3!2!). So the total number we are looking for is

$$\frac{8!}{3!2!} = 3360.$$

(b) It is equivalent to think about this situation in which you need to put 45 **different** balls into 365 different boxes. So the total number of ways is

$$\binom{365}{45} 45! = \frac{365!}{320!} = \texttt{very big, this expression is enough.}$$