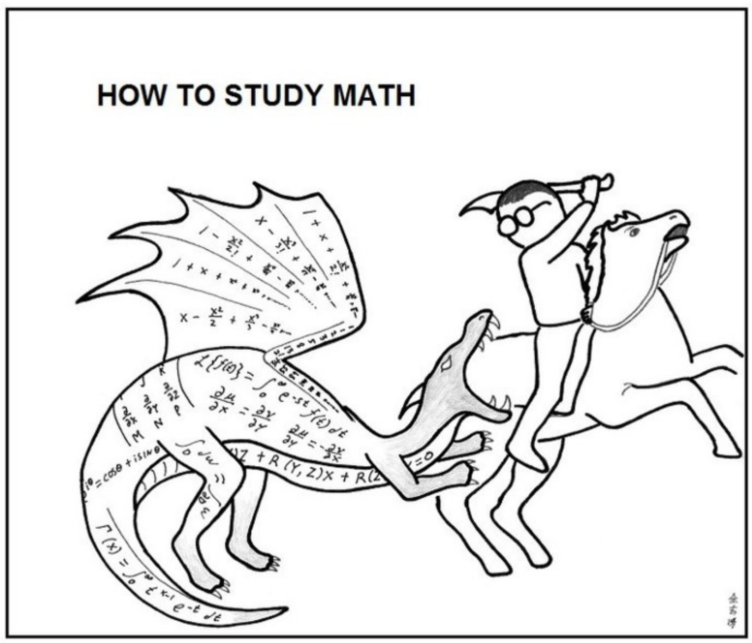
Appendix

In this "*Appendix*", we try to provide some preliminary knowledge for our **Lectures**, which could be more than necessary while the students may want to read when the need arises.



Don't just read it; fight it!

--- *Paul R. Halmos*

1 Basic Set Theory

This section follows closely *Halmos*'s book, as he puts in the *Preface*, "Every mathematician agrees that every mathematician must know some set theory ...... if you want to be a mathematician, you need some, and here it is; read it, absorb it, and forget it." For the purpose of complementing our **Lecture notes**, we would only need to manipulate the ***set operations***, so for example, the ***Partial Ordering, Axiom of Choice, Zorn's Lemma, Schröder-Bernstein's Theorem*** are not covered here. We send interested students to *Halmos*' book.

$\star\star\star$**Reference.** Paul R. Halmos, *Naive Set Theory*, Graduate Texts in Mathematics, Springer.

A group of students, a basket of eggs, or a list of all possible results obtained from an experiment, are all examples of sets of things. To avoid terminological monotony, we sometimes say *collection*, *class* or *space* instead of *set*. The things that a ***set*** contains are called ***members*** or ***elements***. Usually, we use capital letter A, B, *etc*, to denote the sets while we write lower-case ones for the elements. For example, $A = \{a, b, c\}$ means A is a set of three elements a, b, c. If a set contains nothing, *i.e.* no element, we call it an ***empty set***, denoted by $\emptyset$. For a nonempty set A, we write $a \in A$ to indicate a is an element in the set A or a belongs to

A, where $\in$ is the fifth letter of Greek alphabet. One should also note that $\{1,3\} = \{3,1\}$, that is, *the order of elements in a set does not matter.*

Definition 1.1. Given two nonempty sets A and B, we say A is contained in B or B contains A if for any $a \in A$, $a \in B$. And we call A is a subset of B and write $A \subset B$ or $B \supset A$. Moreover, A is called a ***proper subset*** of B if in addition, $A \neq B$, in this case we write $A \subsetneq B$.

Notation. We write $\forall$ to denote "*for any*" or "*for all*" ; $A \subset B$ if $\forall a \in A$, $a \in B$.

Remark 1.2. One can see that $\emptyset$ is a subset of any set; $A = B$ if and only if $A \subset B$ and $B \subset A$. Please note that this is a basic principle to show the equality between two sets. ◀

One can get a subset from a given set A by specifying some certain condition or property, for example $\mathbb{R} = (-\infty, +\infty)$ is the set of real numbers, we define a subset B of $\mathbb{R}$ as follows

$$B = \left\{ x \in \mathbb{R} : \ |x| \leq 1 \ \right\} ,$$

that is, we collect the elements $x \in \mathbb{R}$ that satisfy the condition "*absolute value of x is not greater than 1*" to form a new set B. Clearly, $B = [-1, 1]$. Usually the "*condition*" could be simply a sentence.

Definition 1.3. Given a collection $\mathscr{C}$ of sets, there exists a set Y such that if $x \in X$ for some $X \in \mathscr{C}$, then $x \in Y$ and if $x \in Y$, then $x \in X$ for some $X \in \mathscr{C}$. Such a set Y is called the ***union*** of the collection $\mathscr{C}$. And we write

$$Y = \bigcup_{X \in \mathscr{C}} X \ .$$

Convention. If $\mathscr{C} = \emptyset$, $Y = \emptyset$.

Example 1.4. Let $A = \{1, 2, \text{apple}\}$ and $B = \{1, \text{ strawberry } , a \ , b\}$, then the union of A and B, denoted by $A \cup B$, is equal to $\{1, 2, \text{apple, strawberry } , a, \ b\}$.

Attention! The number 1 is very different from the set $\{1\}$. Think about the following analogy: a box that contains a hat and nothing else is not the same thing as a hat. So similarly, $\{1\} \neq \{\{1\}\}$, the latter is a large box that contains a smaller box with a hat inside (the smaller box)!

Remark 1.5. $\emptyset \cup A = A$ for any set A. In general, for two given sets A, B, $A \subset B$ if and only if $A \cup B = B$. ◀

Definition 1.6. Given a collection $\mathscr{C}$ of sets, there exists a set Z such that

- $x \in Z$ if and only if $x \in X$ for each $X \in \mathscr{C}$; or
- $Z = \emptyset$ and sets in $\mathscr{C}$ have no common elements.

In other words, Z is a set of common elements of sets in $\mathscr{C}$. Such a set Z is called the ***intersection***[1] of the collection $\mathscr{C}$. And we write

$$Z = \bigcap_{X \in \mathscr{C}} X \ .$$

Remark 1.7. To continue with *Example 1.4.*, $A \cap B = \{1\}$. To continue with *Remark 1.5.*, we have $A \cap \emptyset = \emptyset$; $A \subset B$ if and only if $A \cap B = A$. ◀

[1] The existence of intersection, as well as the union is stated as an **axiom**.

Here are some easily proved facts about unions and intersections of sets:

- $A \cup B = B \cup A$; $A \cap B = B \cap A$.
- $A \cup \big(B \cup C\big) = \big(A \cup B\big) \cup C$; $A \cap \big(B \cap C\big) = \big(A \cap B\big) \cap C$.
- $A \cup A = A$; $A \cap A = A$.
- $A \cap \big(B \cup C\big) = \big(A \cap B\big) \cup \big(A \cap C\big)$; $A \cup \big(B \cap C\big) = \big(A \cup B\big) \cap \big(A \cup C\big)$.

Definition 1.8. If A and B are two sets, the ***relative complement*** of B in A or ***set difference*** of A in B is the set $A \setminus B$ defined by

$$A \setminus B = \big\{x \in A \,:\, x \notin B\ \big\}\ .$$

Notation. We also write exchangeably $A - B$ for $A \setminus B$. $x \notin B$ means that x does not belong to B.

Now we consider a *universal* space U, we call $A^c := U \setminus A$ the ***complement*** of A in U, where $A \subset U$.

Here are some easily proved facts about complementation and set difference: If A, B, $C \subset U$,

- $\big(A^c\big)^c = A$; $\emptyset^c = U$; $U^c = \emptyset$; $A \cap A^c = \emptyset$; $A \cup A^c = U$.
- $A - B = A \cap B^c$; $A \subset B$ if and only if $A - B = \emptyset$.
- $\big(A - B\big) - C = \big(A - C\big) - B$ but in general, $\big(A - B\big) - C \neq A - \big(B - C\big)$; $A - \big(A - B\big) = A \cap B$.
- $A \subset B$ if and only if $B^c \subset A^c$; $A \cap \big(B - C\big) = \big(A \cap B\big) - \big(A \cap C\big)$.
- $A \cap \big(B \cup C\big) = \big(A \cap B\big) \cup \big(A \cap C\big)$; $A \cup \big(B \cap C\big) = \big(A \cup B\big) \cap \big(A \cup C\big)$. A generalization of these two equalities is called the ***De Morgan's Law***: let $\big(A_i, i \in I\big)$ be a collection of sets, B be a set, then

$$B \cup \left(\bigcap_{i \in I} A_i\right) = \bigcap_{i \in I} \big(B \cup A_i\big)\ ; \quad B \cap \left(\bigcup_{i \in I} A_i\right) = \bigcup_{i \in I} \big(B \cap A_i\big)\ .$$

Note that there facts can be easily proved and we suggest students do the exercises instead of simply memorizing them. One can use the *Venn Diagram* to grasp some intuition, for instance, see below :